A quasi-polynomial bound for the excluded minors for a surface

Sarah Houdaigoui[1]   Ken-ichi Kawarabayashi[2]

[1]National Institute of Informatics, SOKENDAI

[2]National Institute of Informatics, University of Tokyo

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2. Definitions and preliminary results
3. Structural results: Forbidden structures
4. Main proof
   - Bounding the degree of G and the maximum size of a face of (G, Π)
   - Bounding the height of a tree decomposition of G
   - Putting everything together
5. Conclusion

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Definition of minor and excluded minor

Definition (Minor)

A minor H of a graph G can be obtained from G by a series of vertex deletions, edge deletions and edge contractions.

Definition (Excluded minor)

Let $\mathcal{C}$ be a class of graphs. An excluded minor for the class $\mathcal{C}$ is a graph $G \notin \mathcal{C}$ so that every proper minor of G is in $\mathcal{C}$.

Definition of surface, embedding and genus

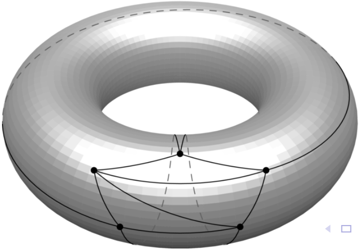
Definition

A surface is a connected compact Hausdorff topological space which is locally homeomorphic to an open disc in the plane.

Embedding (informal definition): An embedding Π of a graph G on a surface S is a drawing of G on S without crossings.

Genus: Euler genus (measure of the complexity of a surface)

Examples: Sphere (g=0), torus (g=2), double-torus (g=4), projective plane (g=1), Klein bottle (g=2)...



The Graph Minor theorem

Theorem (Robertson & Seymour [4])

Every family of graphs that is closed under minors can be defined by a finite set of forbidden minors.

Corollary (Robertson & Seymour [3])

Let S be a surface. Let $\mathcal{C}_S$ be the class of graphs that can be embedded on S without crossings. Then there is a finite number of excluded minors for $\mathcal{C}_S$.

The Graph Minor theorem

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Theorem (Wagner)

A graph is planar if and only if it does not contain K_5 or $K_{3,3}$ as its minor.

A bound on the size of these excluded minors

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Theorem (Seymour 1993 [5])

Let S be a given surface of genus g, every excluded minor for S has at most 2^{2^k} vertices where $k = (3g+9)^9$.

Main result: a quasi-polynomial bound

Theorem (H., Kawarabayashi 2025+)

Let S be a given surface of Euler genus g. Every excluded minor for S has at most $U(g) = O(g^{\log^3 g})$ vertices.

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Let S be a given surface of Euler genus g. Every excluded minor for S has at most $U(g) = O(g^{\log^3 g})$ vertices.

Conjecture

Let S be a given surface of genus g, every excluded minor for S has a number of vertices polynomial in g.

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Contractible and homotopic cycles

Let G be a Π-embedded graph in a surface S.

Definition (Contractible cycle)

Let C be a cycle of G, C is Π-contractible if C bounds a disk in the embedding Π of G.

Definition (Homotopic cycles)

Let C, C' be two cycles of G, C and C' are Π-homotopic if $C \cup C'$ bound a (degenerate) cylinder in the embedding Π of G.

Treewidth and tree decomposition

The treewidth is a graph parameter that measures how close a graph is to a tree.

Definition (Tree decomposition)

A tree decomposition of a graph G is a pair $(T, (V_t)_{t \in V(T)})$ with T a tree and, for every $t \in V(T)$, $V_t \subseteq V(G)$ with the following properties:

- $\bigcup_{t \in V(T)} V_t = V(G)$,
- *for every $e = uv \in E(G)$, there exists $t \in V(T)$ so that $u, v \in V_t$,*
- *for $t, t', t'' \in V(T)$ so that t' is on the path between t and t'' in T, $V_t \cap V_{t''} \subseteq V_{t'}$.*

The width of a tree decomposition $(T, (V_t)_{t \in V(T)})$ of G is $\max_{t \in V(T)} |V_t| - 1$ and the treewidth of G is the minimal width of its tree decompositions.

Folklore on surfaces and connectivity

Lemma

Let $H_1, ..., H_p$ ($p \geq 1$) be the 2-connected blocks of a graph H, then

$$g(H) = g(H_1) + ... + g(H_p)$$

Lemma

Let G be an excluded minor for a surface S of genus g. Let $G_1, ..., G_p$ ($p \geq 1$) be the 2-connected blocks of G. Then, for $1 \leq i \leq p$, G_i is an excluded minor for some surface S_i.

A result on connectivity

Lemma

Let G be an excluded minor for a surface S of genus g. Suppose that, for any 2-connected graph H that is an excluded minor for some surface S_H, $|V(H)| \leq N(g(S_H))$ with N an increasing function.
Then, $|V(G)| \leq (g+2) \times N(g)$.

A result on connectivity

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From now on: Let S, S' be surfaces with S' of Euler genus g and S of Euler genus $g+1$ or $g+2$. Let G be a 2-connected excluded minor for the surface S' and suppose that G can be embedded in surface S with embedding Π.

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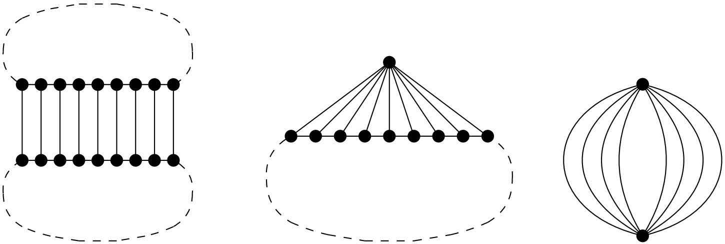
Isolated paths

We define a *piece* as a vertex or a face of (G, Π).

Proposition (H., Kawarabayashi 2025+)

G contains at most $4 \times (6g(\Pi) - 5) \leq 4 \times (6g + 7)$ isolated paths in Π from a piece p to a piece p'.



(a) Disjoint isolated paths

(b) Almost disjoint isolated paths

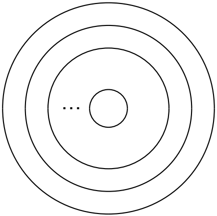
(c) Joint isolated paths

Figure: Isolated paths. The solid lines indicate paths, whereas the dotted lines show the boundaries of the faces which the isolated paths use.

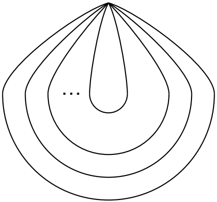
Well-nested cycles

Proposition (H., Kawarabayashi 2025+)

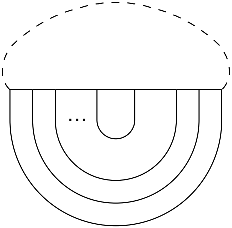
Let $q = \frac{1153}{1152}$ and $m = 2(\lfloor \log_q(3g+4) \rfloor + 2)$. The graph G contains at most m cycles that are Π-well-nested.



(a) Fully well-nested cycles



(b) Well-nested cycles pinched on a vertex



(c) Well-nested cycles pinched on a face

Figure: Well-nested cycles. The solid lines indicate paths, whereas the dotted lines show the boundaries of the faces which the isolated paths use.

Known results on tree decompositions of G

Theorem (Seymour [5, (3.3)])

The treewidth of G is bounded by a polynomial in g:

$$tw(G) \leq T(g)$$

with $T(g) = 3(g+3)^2(3g+16) - 3 = O(g^3)$

Theorem (Seymour [5, claim (5) in (4.1)])

Let $(T, (V_t)_{t \in T})$ be a tree decomposition of G of width $< w$. Then, the maximum degree of T is bounded by a polynomial in g and w:

$$\Delta(T) \leq \Delta_T(g, w)$$

with $\Delta_T(g, w) = 2g + 2w$

First consequence: treewidth

Corollary (H., Kawarabayashi 2025+)

The treewidth of G is bounded by the following function of g:

$$tw(G) \leq T(g)$$

with $T(g) = 264(g+2)(m+1) - 1 = O(g \log g)$*, where* $m = 2(\lfloor \log_q(3g+4) \rfloor + 2)$ *and* $q = \frac{1153}{1152}$.

Corollary (H., Kawarabayashi 2025+)

Let $(T, (V_t)_{t \in T})$ *be a tree decomposition of G of width* $tw(G)$*. Then, the degree of T is bounded by a polynomial in g:*

$$\Delta(T) \leq \Delta_T(g)$$

with $\Delta_T(g) = \Delta_T(g, T(g)+1) = 2g + 2(T(g)+1) = O(g \log g)$.

Second consequence: bound on the size of an excluded grid

Theorem (Thomassen [6])

Let G be a 2-connected excluded minor for a surface of Euler genus g. Then G contains no subdivision of the $4k \times 2k$ grid, with $k = \lceil 800g^{3/2} \rceil$.

Corollary (H., Kawarabayashi 2025+)

Let G be a 2-connected excluded minor for a surface of Euler genus g. Then G contains no subdivision of the $4k \times 2k$ grid, with $k = O(\sqrt{g} \log g)$.

Well-homotopic cycles

Proposition (H., Kawarabayashi 2025+)

Let $q = \frac{1153}{1152}$ *and* $m = 2(\lfloor \log_q(3g+4) \rfloor + 2)$*. G contains at most* $2m$ Π*-well-homotopic cycles.*

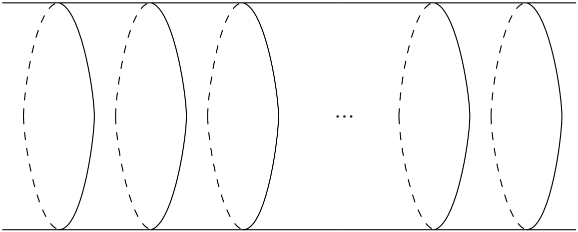


Figure: Well-homotopic cycles.

Consequence of the well-homotopic cycles theorem

Corollary (H., Kawarabayashi 2025+)

Let $q = \frac{1153}{1152}$ *and* $m = 2(\lfloor \log_q(3g+4) \rfloor + 2)$*.* G *contains at most* $2m \times (3g+3) = O(g \log g)$ *disjoint* Π*-noncontractible cycles.*

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Bounding the degree of G and the maximum size of a face of (G, Π)

Theorem (H., Kawarabayashi 2025+)

Let $\tilde{g} = 4(6g+7)$, $q = \frac{1153}{1152}$ *and* $m = 2(\lfloor \log_q(3g+4) \rfloor + 2)$.

$$\Delta(G) \leq \Delta(g) \quad \textit{and} \quad \Delta_F(G, \Pi) \leq \Delta(g)$$

with $\Delta(g) = 2m(\tilde{g}+1)^4 \left(4m(\tilde{g}+1)^2\right)^{m^2} = O(g^{\log^2 g})$

Bounding the degree of G and the maximum size of a face of (G, Π)

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Proof outline: Prove by induction that G contains $m+1$ Π-well-nested cycles. Contradiction.

Bounding the height of a tree decomposition of G

Proposition (H., Kawarabayashi 2025+)

Let $(T, (V_t)_{t\in T})$ be a (nice) tree decomposition of G of width w. Let P be a path from t_1 to t_2 of length $P(g, w)$ in T with

$$P(g,w) = \frac{\Delta(g)(\Delta(g)^{2m}-1)}{\Delta(g)-1} \times 2w + w + 2 = O(g^{\log^3 g} \times w)$$

Let $G_0 = \bigcup_{t\in \overline{P}} V_t - (V_{t_1} \cup V_{t_2})$. Then $\Pi(G_0)$ is not an embedding in a disk on S.

Proof outline: Proceed by contradiction: G_0 is in a disk on S. Use the bound on the number of nested cycles and the separators given by the tree decomposition to prove a bound on the number of vertices of G_0.

Bounding the height of a tree decomposition of G

Theorem (H., Kawarabayashi 2025+)

Let $(T, (V_t)_{t \in T})$ be a tree decomposition of G of width $tw(G)$. Then, T contains no path of length more than $P'(g, w) = (2m(3g+3)+1) \times P(g, w) - 1 = O(g^{\log^3 g} \times w)$.

Proof outline: Proceed by contradiction: there is a path of length $> P'(g, w)$. Cut this path into paths of length $\geq P(g, w)$, there are at least $2m(3g+3)+1$ of them. Contradiction.

Recap of the results regarding tree decomposition

- Treewidth of G: $O(g \log g)$
- Maximum degree of the tree of an optimal tree decomposition of G: $O(g \log g)$
- Height of an optimal tree decomposition of G: $O(g^{\log^3 g})$

Recap of the results regarding tree decomposition

- Treewidth of G: $O(g \log g)$
- Maximum degree of the tree of an optimal tree decomposition of G: $O(g \log g)$
- Height of an optimal tree decomposition of G: $O(g^{\log^3 g})$

→ There is an obvious bound on the order of the tree and therefore on the order of G.

A quasi single-exponential bound for G

Corollary (H., Kawarabayashi 2025+)

Let G be an excluded minor for a surface S' of genus g.

$$|V(G)| \leq 2^{Q(g)}$$

with $Q(g)$ a quasi-polynomial in g so that

$$Q(g) \geq \log((T(g)+1) \times \Delta(g)^{P(g)})$$

From a quasi single-exponential to a quasi polynomial bound: pathwidth

Trick: Switch to pathwidth

From a quasi single-exponential to a quasi polynomial bound: pathwidth

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Proposition (Bodlaender [1])

Let G be a graph, then

$$pw(G) = O(tw(G)\log(|V(G)|))$$

From a quasi single-exponential to a quasi polynomial bound: pathwidth

Trick: Switch to pathwidth

Proposition (Bodlaender [1])

Let G be a graph, then

$$pw(G) = O(tw(G)\log(|V(G)|))$$

Corollary (H., Kawarabayashi 2025+)

Let G be an excluded minor for a surface S of genus g. There exists a constant A so that

$$pw(G) \leq A \times T(g) \times Q(g)$$

A quasi-polynomial bound

Corollary (H., Kawarabayashi 2025+)

Let G be an excluded minor for a surface S of genus g. There exists a constant A so that

$$|V(G)| \leq A \times S(g)$$

with $S(g) = P(g) \times T(g) \times Q(g) = O(g^{\log^3 g})$

Proof outline: Use the bound on the pathwidth and use again the bound on the height of the tree in the tree decomposition (= size of the path).

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Conclusion: From a double-exponential to a polynomial bound

Theorem (Seymour 1993 [5])

Let S be a given surface of Euler genus g. Every excluded minor for S has at most 2^{2^k} vertices where $k = (3g+9)^9$.

Theorem (H., Kawarabayashi 2025+)

Let S be a given surface of Euler genus g. Every excluded minor for S has at most $U(g) = O(g^{\log^3 g})$ vertices.

Conclusion: Subsidiary results

- Forbidden structures: isolated paths, nested cycles, homotopic cycles
- Treewidth:
$$O(g^3) \to O(g \log g)$$
- Maximum degree of the tree of an optimal tree decomposition of G:
$$O(g^3) \to O(g \log g)$$
- Maximum size of a subdivision of a grid in G:
$$O(g^{3/2}) \to O(\sqrt{g} \log g)$$

Future work

We are currently pursuing research in order to show a polynomial bound on the order of G.

Conjecture

Let S be a given surface of genus g, every excluded minor for S has a number of vertices polynomial in g.

Thank you for your attention

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