A polynomial bound for the excluded minors for a surface

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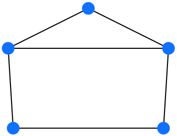
1. Introduction

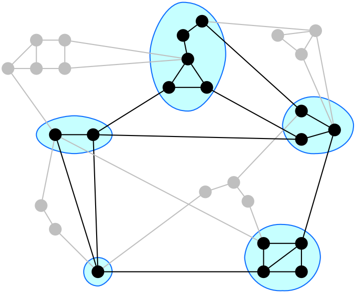
Definition of minor

Definition (Minor)

A minor H of a graph G can be obtained from G by a series of vertex deletions, edge deletions and edge contractions.



(a) The house graph H



(b) Model of H in a graph G

Figure: Minor

Definition of surface and embedding

Examples of surfaces: Sphere (g=0), torus (g=2), double-torus (g=4), projective plane (g=1), Klein bottle (g=2)...

Embedding (informal definition): An embedding Π of a graph G on a surface S is a drawing of G on S without crossings.

Genus: Measure of the complexity of a surface (Euler genus)

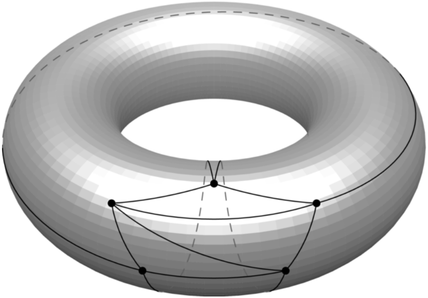


Figure: An embedding of K_5 on the torus

Family of graphs closed under minors

Definition (Closed under minors)

A family of graphs $\mathcal{C}$ is closed under minors if, for every $G \in \mathcal{C}$ and H minor of G, we have $H \in \mathcal{C}$.

Definition (Excluded minor)

Let $\mathcal{C}$ be a class of graphs closed under minors. An excluded minor for the class $\mathcal{C}$ is a graph $G \notin \mathcal{C}$ so that every proper minor of G is in $\mathcal{C}$.

Notice that: *G is minimal so that $G \notin \mathcal{C}$*

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Theorem (Graph Minor Theorem, Robertson & Seymour 2004)

Every family of graphs that is closed under minors can be characterized by a finite set of excluded minors.

The Graph Minor Theorem for graphs on surfaces

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Lemma

Let S be a surface. Let $\mathcal{C}_S$ be the class of graphs that can be embedded on S without crossings. Then $\mathcal{C}_S$ is closed under minors.

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Theorem (Wagner, also corollary of the GMT)

A graph is planar if and only if it does not contain K_5 or $K_{3,3}$ as its minor.

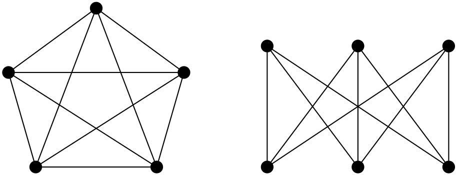

Figure: The excluded minors for the sphere: K_5 and $K_{3,3}$

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- **For the projective plane:** exactly 35 excluded minors, explicitly known (GH78, GHW79, A81)
- **For the torus:** more than 2200 excluded minors, maybe a lot more (DH72, D78, BW86, J95, H97)

A bound on the excluded minors for a surface

Theorem (Seymour 1993)

Let S be a given surface of genus g, every excluded minor for S has at most 2^{2^k} vertices where $k = (3g+9)^9$.

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Algorithmic implications

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There are several algorithms that relies on bounds for the excluded minors for surfaces:

- Membership test for graphs on surfaces (FL89, AGK08)
  order or *treewidth*
- Embedding graphs in a given surface (KMR08) *order*
- Computing a graph minor decomposition (GKR13) *order*

2. Preliminary results

Genus of a graph

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The genus of a graph G is the genus of the smallest surface in which G can be embedded.

- It is well defined: There is always a surface in which G can be embedded.
- It is minor-monotone: if H is a minor of G, then $g(H) \leq g(G)$.

Excluded minor for a surface

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  Let $e \in E(G)$, embed $G - e$ in the surface S with embedding Π_{G-e}. Then, adding the edge e to the embedding Π_{G-e} (in any way) create an embedding Π in a surface of genus $g+1$ or $g+2$.

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- G is 2-connected.

  Otherwise, decompose it into its 2-connected blocks.

Lemma

Let $G_1, \ldots, G_p$ ($p \geq 1$) be the 2-connected blocks of G. Then, for $1 \leq i \leq p$, G_i is an excluded minor for some surface S_i.

Lemma

Let $G_1, \ldots, G_p$ ($p \geq 1$) be the 2-connected blocks of G. Then, $g(G) = g(G_1) + \ldots + g(G_p)$.

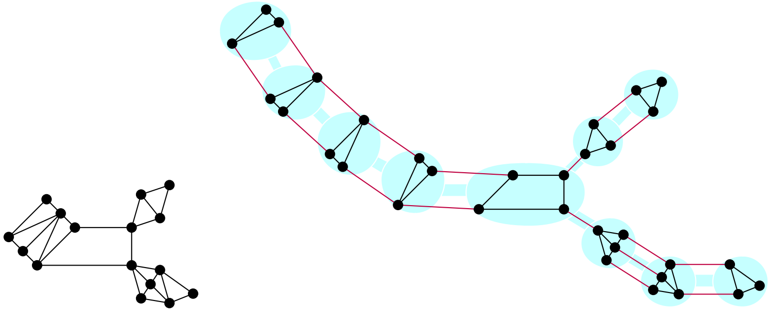
3. Treewidth and graphs on surfaces

Tree decomposition

Definition (Tree decomposition)

A tree decomposition of a graph G is a pair $(T, (V_t)_{t \in V(T)})$ with T a tree and, for every $t \in V(T)$, $V_t \subseteq V(G)$ with the following properties:

- $\forall v \in V(G), \{t \in V(T), v \in V_t\}$ *is a (non empty) tree,*
- $\forall e = uv \in E(G), \exists t \in V(T)$ *so that* $u, v \in V_t$.



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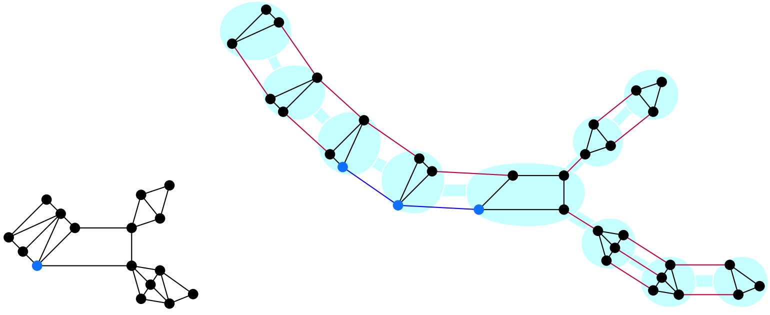

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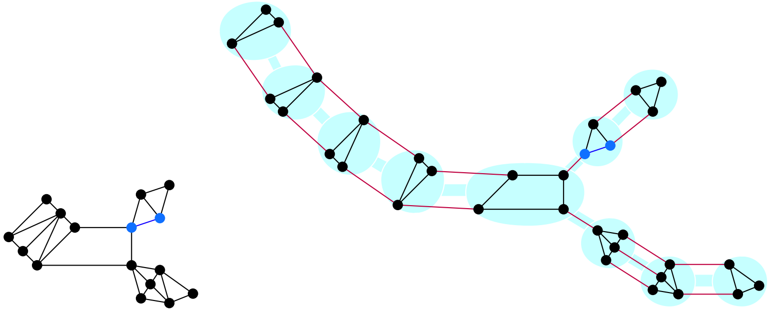

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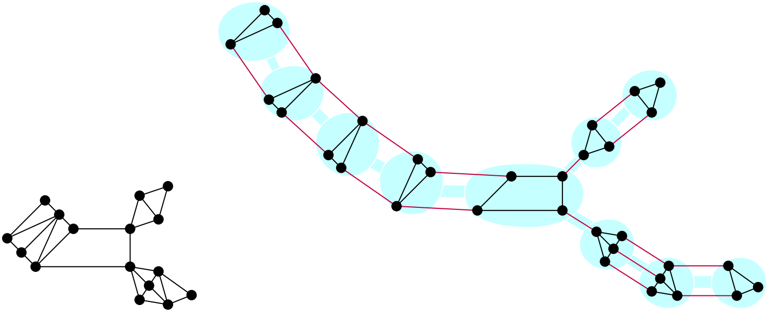
Treewidth

The treewidth is a graph parameter that measures how close a graph is to a tree.

Definition (Width and treewidth)

The width of $(T, (V_t)_{t \in V(T)})$ *of* G *is* $\max_{t \in V(T)} |V_t| - 1$ *and the treewidth of* G *is the minimal width of its tree decompositions.*



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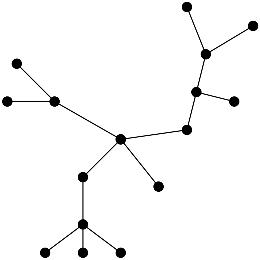
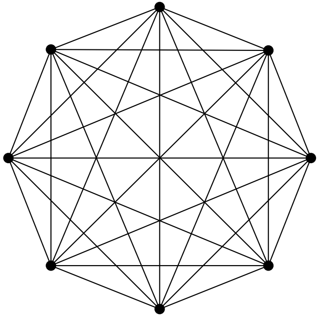


(a) A tree T: $tw(T) = 1$



(b) The clique K_8: $tw(K_8) = 7$

Treewidth and graphs on surfaces

Treewidth and graphs on surfaces

Planar graphs have unbounded treewidth.

Lemma (Treewidth of a grid)

For $k \geq 1$, the $k \times k$ grid has treewidth k.

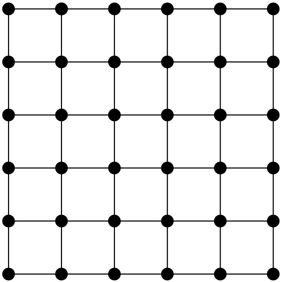


Figure: The 6×6 grid has treewidth 6.

Therefore, for a surface S, graphs embeddable on S have unbounded treewidth.

Tree decompositions of excluded minors for surfaces

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Treewidth of G:

$O(g^3)$	Seymour 1993
$\downarrow$	
$O(g \log g)$	H., Kawarabayashi SODA'26

4. Main structural results

Main structural result: Disjoint nested cycles

G has an embedding Π in a surface S' of genus $g+1$ or $g+2$.

Proposition (H., Kawarabayashi)

Let $q = \frac{9073}{9072}$ and $m = 2(\lfloor \log_q(3g+4) \rfloor + 2)$. The graph G contains at most m disjoint Π-contractible nested cycles.

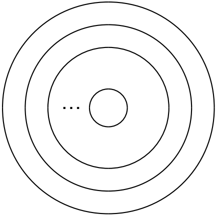

Figure: Disjoint nested cycles.

Contractible: Let H be a Π_H-embedded graph and C be a cycle of H, C is Π_H-contractible if C bounds a disk in the embedding Π_H of H.

Main structural result: Disjoint homotopic cycles

Proposition (H., Kawarabayashi)

Let $q = \frac{9073}{9072}$ and $m = 2(\lfloor \log_q(3g+4) \rfloor + 2)$. The graph G contains at most $2m$ disjoint Π-homotopic cycles.

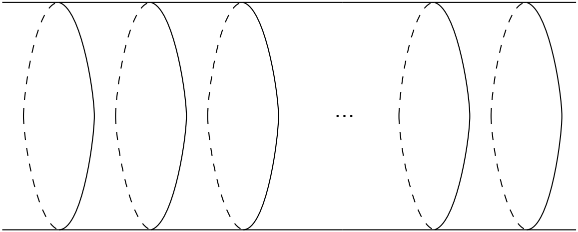

Figure: Disjoint homotopic cycles.

Homotopic: Let H be a Π_H-embedded graph and let C and C' be disjoint cycles of H, C and C' are Π_H-homotopic if they bound a cylinder in the embedding Π_H of H.

The disjoint Π-noncontractible cycles in (G, Π)

Proposition

Let H be a Π_H-embedded graph in a surface of genus g_H. If $C_1, \ldots, C_k$ are cycles of H that are disjoint, Π_H-noncontractible and pairwise Π_H-nonhomotopic, then

$$k \leq \begin{cases} g_H & \text{if } g_H \leq 1 \\ 3g_H - 3 & \text{if } g_H \geq 2 \end{cases}$$

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Corollary (H., Kawarabayashi)

Let $q = \frac{9073}{9072}$ and $m = 2(\lfloor \log_q(3g+4) \rfloor + 2)$. The graph G contains at most

$$2m \times (3g+3) = O(g \log g)$$

disjoint Π-noncontractible cycles.

A separator-based variant of disjoint nested cycles

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Proposition

Let $q = \frac{9073}{9072}$. Let C be a Π-contractible cycle of G and suppose that $A \subseteq V(C)$ separates the interior and the exterior of C with $|A| = k$. Then, Int(C, Π) contains at most $2(\lfloor \log_q(60k + 180) \rfloor + 2)$ disjoint Π-nested cycles.

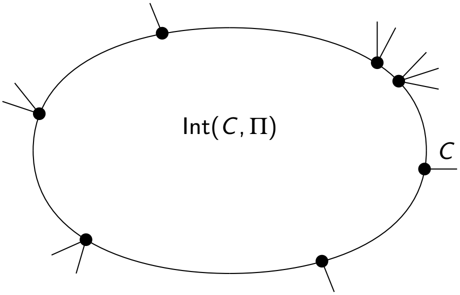

Figure: The cycle C and its separator.

Interest of a separator-based variant

The bound for the disjoint nested cycles is:

- globally

$$O(\log g)$$

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The bound for the disjoint nested cycles is:

- globally

$$O(\log g)$$

  We believe this bound to be essentially tight.

- in a Π-contractible subgraph with a separator of size k

$$O(\log k)$$

If $k << g$, then we get a significant improvement locally.

Proof strategy

Let G be an excluded minor for a surface S of genus g.

To bound the order of G, let's make use of the separator-based variant of disjoint nested cycles.

Goal: Find $G_1 \subseteq G$ such that:

- G_1 is a 2-connected Π-contractible subgraph of G,
- G_1 has a separator of size $O(\log^2 g)$,
- $|V(G)| \leq f(g) \times |V(G_1)|$ for some polynomial function f.

5. Bound on the order of G

Proof outline

1. Step 1: reduce from G to a planar subgraph $G_0 \subseteq G$
2. Step 2: reduce from G_0 to $G_1 \subseteq G_0$ with a $O(\log^2 g)$ separator
3. Step 3: show that the order of G_1 is sub-polynomial in g

Step 1: reduce from G to a planar subgraph $G_0 \subseteq G$

Balanced tree decomposition of G

We use the balanced separator theorem from Robertson and Seymour (RS86)

Balanced tree decomposition of G

We use the balanced separator theorem from Robertson and Seymour (RS86) to show the following:

Lemma

Let H be a graph and let $(T, (V_t)_{t \in T})$ be a tree decomposition of H. There is a sequence of subtrees $(T_1, \ldots, T_k)$ of T such that, for every $1 \leq i, j \leq k$,

1. $|\bigcup_{t \in T_i} V_t| \leq 3|\bigcup_{t \in T_j} V_t|$,
2. $V(T_i) \cap (\bigcup_{1 \leq j \neq i \leq k} V(T_j))$ *has size at most* $\lfloor \log_{\frac{4}{3}} 3k \rfloor$.

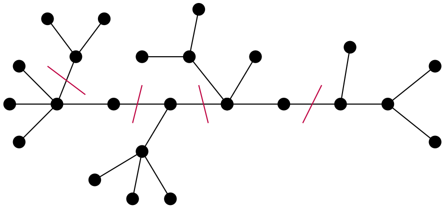

Figure: A tree decomposition with a balanced separation

Reducing from G to a planar subgraph $G_0 \subseteq G$

Take a tree decomposition of G of width $tw(G)$ and divide it into $2m \times (3g+3)+1$ balanced parts.

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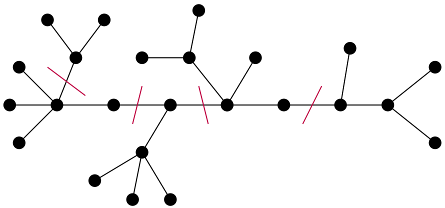

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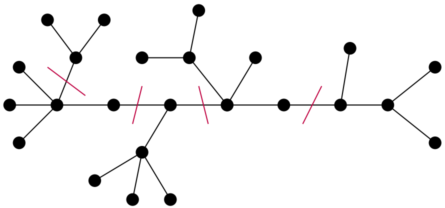

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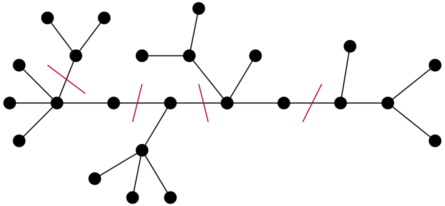

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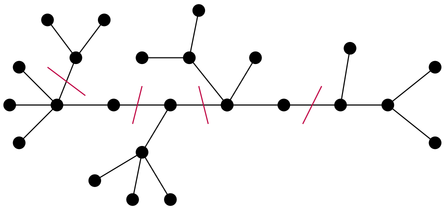

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Using the $2m \times (3g+3)$ bound on the number of disjoint Π-noncontractible cycles in (G, Π), we get that all the connected components induced by one of the parts are Π-contractible.

Let G_0 be the largest connected component. We can bound the number of connected components by $\tilde{O}(g^3)$.

Reducing from G to a planar subgraph $G_0 \subseteq G$

We found:

- G_0 is Π-contractible,
- G_0 has a separator A of size $O(g \log^2 g)$,
- $|V(G)| \leq f(g) \times |V(G_0)|$ with $f(g) = \tilde{O}(g^4)$.

Step 2: reduce from G_0 to $G_1 \subseteq G_0$ with a $O(\log^2 g)$ separator

The treewidth of G_0

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Theorem

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Therefore, G_0 cannot contain a $2(m + 1) \times 2(m + 1)$ grid minor.

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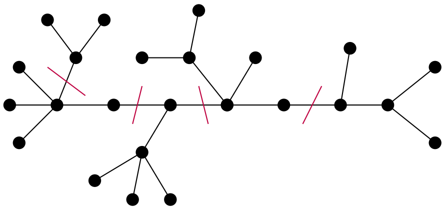
→ G_0 has treewidth at most $12m + 7$ by the result stated above.

Reducing from G_0 to $G_1 \subseteq G_0$ with a $O(\log^2 g)$ separator

Take a tree decomposition of G_0 of width at most $12m + 7$ and divide it into $3|A| + 1 = O(g \log^2 g)$ balanced parts.



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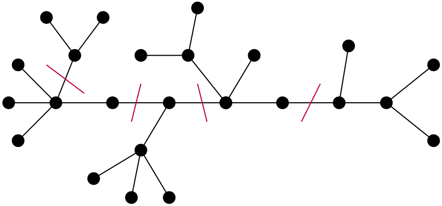

Figure: A tree decomposition with a balanced separation

Using that G_0 is planar, we get that the connected components induced by one of the parts contain each at most 3 vertices in A.

Reducing from G_0 to $G_1 \subseteq G_0$ with a $O(\log^2 g)$ separator

Take a tree decomposition of G_0 of width at most $12m + 7$ and divide it into $3|A| + 1 = O(g \log^2 g)$ balanced parts.

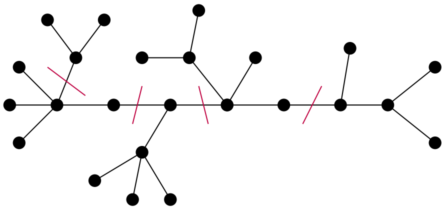

Figure: A tree decomposition with a balanced separation

Using that G_0 is planar, we get that the connected components induced by one of the parts contain each at most 3 vertices in A.

Let G_1 be the largest 2-connected component of this part. We can bound the number of 2-connected components by $\tilde{O}(g^3)$.

Reducing from G_0 to $G_1 \subseteq G_0$ with a $O(\log g)$ separator

We found:

- G_1 is Π-contractible and 2-connected,
- G_1 has a separator A of size $O(\log^2 g)$,
- $|V(G)| \leq f(g) \times |V(G_1)|$ with $f(g) = \tilde{O}(g^8)$.

Step 3: show that the order of G_1 is sub-polynomial in g

Showing that the order of G_1 is sub-polynomial in g

- Step 3.1: bound the maximum degree in G_1 and the maximum size of faces in (G_1, Π)
- Step 3.2: bound the order of G_1

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If G_1 has a large degree vertex:

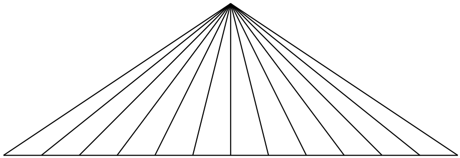

Figure: Faces on a large degree vertex.

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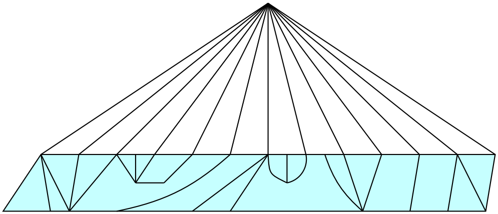

Figure: Faces on a large degree vertex.

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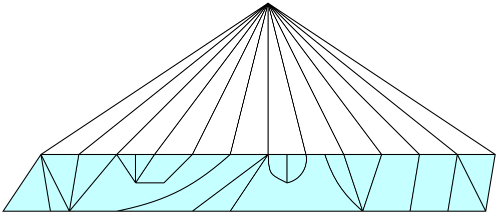

Figure: Faces on a large degree vertex.

Make use of the $O(\log \log g)$ bound on the number of Π-nested cycles.

Step 3.2: bound the order of G_1

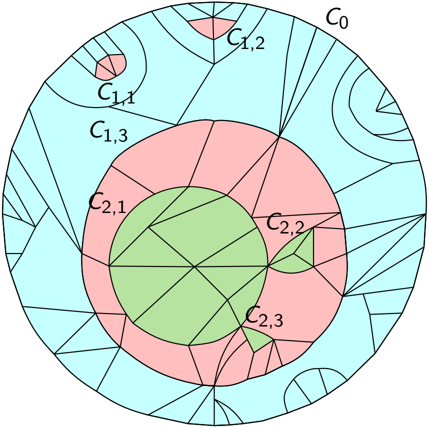

Figure: Partition into radius classes with respect to C_0.

Radius (of a face): Distance of this face to the outer face.

6. Conclusion

Conclusion: From a double-exponential to a polynomial bound

Theorem (Seymour 1993)

Let S be a given surface of genus g, every excluded minor for S has at most 2^{2^k} vertices where $k = (3g+9)^9$.

→ From a double-exponential bound...

Theorem (H., Kawarabayashi)

Let S be a given surface of Euler genus g. Every excluded minor for S has $O(g^{8+\varepsilon})$ vertices for every $\varepsilon > 0$.

→ ... to a polynomial bound

Future work

We conjecture that the optimal bound is either almost linear or almost quadratic.

Open problem

Is there an almost linear bound on the order of G?

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Is there an almost quadratic bound on the order of G? More specifically, is the optimal bound almost quadratic?

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Nevertheless, we believe that achieving an exponent below 5 will require fundamentally new techniques.