An almost quadratic bound for the minimal excluded minors for a surface

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July 22, 2026

A characterization of the planar graphs

Theorem (Wagner, 1937)

A graph is planar if and only if it does not contain K_5 or $K_{3,3}$ as its minor.

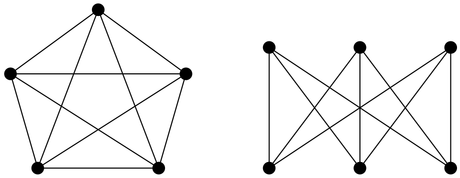


Figure: The graphs K_5 and $K_{3,3}$

A characterization of the graphs embeddable in a surface

Question

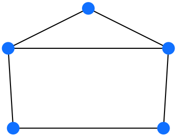
Let S be a surface. Does there exist a characterization of graphs embeddable in S similar to Wagner theorem for planar graphs?

1. Introduction

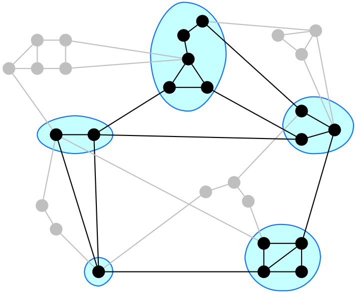
Definition of minor

Definition (Minor)

A minor H of a graph G can be obtained from G by a series of vertex deletions, edge deletions and edge contractions.



(a) The house graph H



(b) Model of H in a graph G

Figure: Minor

Definition of surface and embedding

Examples of surfaces: Sphere (g=0), torus (g=2), double-torus (g=4), projective plane (g=1), Klein bottle (g=2)...

Embedding (informal definition): An embedding Π of a graph G on a surface S is a drawing of G on S without crossings.

Genus: Measure of the complexity of a surface (Euler genus)

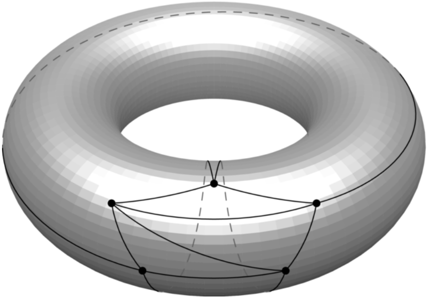


Figure: An embedding of K_5 on the torus

Class of graphs closed under minors

Definition (Closed under minors)

A class of graphs $\mathcal{C}$ is closed under minors if, for every $G \in \mathcal{C}$ and H minor of G, we have $H \in \mathcal{C}$.

Definition (Excluded minor)

Let $\mathcal{C}$ be a class of graphs closed under minors. An excluded minor for the class $\mathcal{C}$ is a graph $G \notin \mathcal{C}$ so that every proper minor of G is in $\mathcal{C}$.

Notice that: G *is minimal so that* $G \notin \mathcal{C}$

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Theorem (Graph Minor Theorem, Robertson & Seymour 2004)

Every class of graphs that is closed under minors can be characterized by a finite set of excluded minors.

The Graph Minor Theorem for graphs on surfaces

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Lemma

Let S be a surface. Let $\mathcal{C}_S$ be the class of graphs that can be embedded on S without crossings. Then $\mathcal{C}_S$ is closed under minors.

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Theorem (Wagner, also corollary of the GMT)

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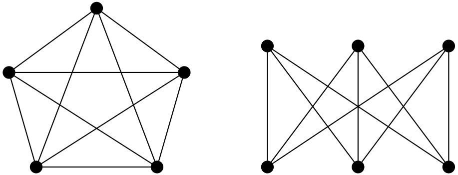

Figure: The excluded minors for the sphere: K_5 and $K_{3,3}$

Research inspired by the Graph Minor Theorem

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- Finding the excluded minors: bounded pathwidth (K92, TUK94), projective plane (GH78, GHW79, A81), torus (DH72, BW86, NM97, GMC09, MS14, MW18...), Klein bottle (MS24) ...

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- Improvement on the graph minor theorem: shorter proofs, improved bounds for various steps (polynomial bounds)

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- Algorithms based on the excluded minors for a given minor-closed class. Examples: embedding in a surface (KMR08), graph minor decomposition (GKR13)

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- **For the projective plane:** exactly 35 excluded minors, explicitly known (GH78, GHW79, A81)
- **For the torus:** more than 17,500 excluded minors, maybe a lot more (DH72, D78, BW86, N93, J95, NM97, H97, C02, GMC09, MS14, MW18)

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Theorem (Seymour 1993)

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- Useful to computational approaches aimed at finding the excluded minors of small-genus surfaces such as the torus and the Klein bottle (NM97, MW18)
- Make the qualitative results from the graph minor theorem efficient, by finding polynomial bounds (CC16, KTW20, GSW25)

2. Preliminary results

Genus of a graph

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The genus of a graph G is the genus of the smallest surface in which G can be embedded.

- It is well defined: There is always a surface in which G can be embedded.
- It is minor-monotone: if H is a minor of G, then $g(H) \leq g(G)$.

Excluded minor for a surface

Take G to be an excluded minor for a surface S of genus g. What can we say?

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  Let $e \in E(G)$, embed $G-e$ in the surface S with embedding Π_{G-e}. Then, adding the edge e to the embedding Π_{G-e} (in any way) create an embedding Π in a surface of genus $g+1$ or $g+2$.

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- We can suppose that G is 2-connected.

  Otherwise, decompose it into its 2-connected blocks.

Lemma

Let $G_1, \ldots, G_p$ ($p \geq 1$) be the 2-connected blocks of G. Then, for $1 \leq i \leq p$, G_i is an excluded minor for some surface S_i.

Lemma

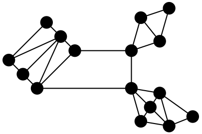
Let $G_1, \ldots, G_p$ ($p \geq 1$) be the 2-connected blocks of G. Then, $g(G) = g(G_1) + \ldots + g(G_p)$.

3. Treewidth and graphs on surfaces

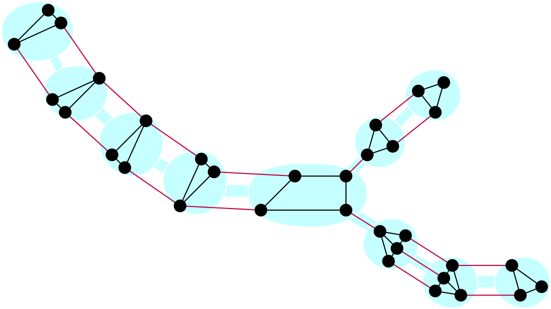
Tree decomposition

Definition (Tree decomposition (informal))

A tree decomposition of a graph G is a pair $(T, (V_t)_{t \in V(T)})$ with T a tree and, for every $t \in V(T)$, $V_t \subseteq V(G)$ with specific properties.



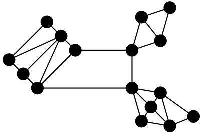
(a) A graph G



(b) A tree decomposition of G

Tree decomposition

Key property: In a tree decomposition of G, the intersection of two adjacents bags V_t and $V_{t'}$ is a separator for G.



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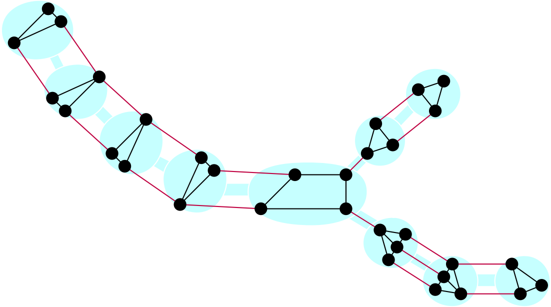


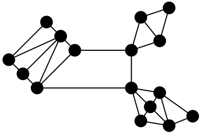
(b) A tree decomposition of G

Treewidth

The treewidth is a graph parameter that measures how close a graph is to a tree.

Definition (Width and treewidth)

The width of $(T, (V_t)_{t \in V(T)})$ *of* G *is* $\max_{t \in V(T)} |V_t| - 1$ *and the treewidth of* G *is the minimal width of its tree decompositions.*



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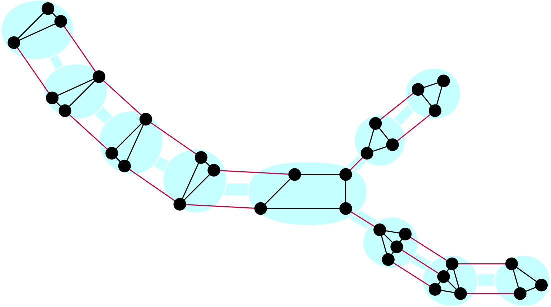

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Treewidth and graphs on surfaces

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Planar graphs have unbounded treewidth.

Lemma (Treewidth of a grid)

For $k \geq 1$, the $k \times k$ grid has treewidth k.

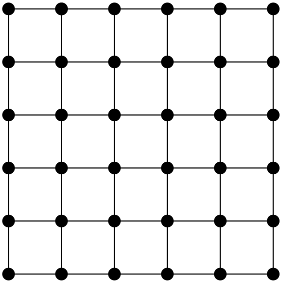

Figure: The 6×6 grid has treewidth 6.

Therefore, for a surface S, graphs embeddable on S have unbounded treewidth.

Treewidth of the excluded minors for surfaces

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Treewidth of G:

$O(g^3)$	Seymour 1993
$\downarrow$	
$O(g \log g)$	H., Kawarabayashi

4. Main structural results

Main structural result: Disjoint nested cycles

G has an embedding Π in a surface S' of genus $g+1$ or $g+2$.

Proposition (H., Kawarabayashi)

Let $q = \frac{9073}{9072}$ and $m = 2(\lfloor \log_q(3g+4) \rfloor + 2)$. The graph G contains at most m disjoint Π-contractible nested cycles.

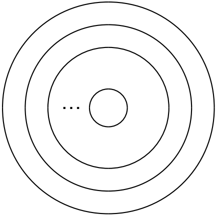

Figure: Disjoint nested cycles.

Contractible: Let H be a Π_H-embedded graph and C be a cycle of H, C is Π_H-contractible if C bounds a disk in the embedding Π_H of H.

Main structural result: Disjoint homotopic cycles

Proposition (H., Kawarabayashi)

Let $q = \frac{9073}{9072}$ and $m = 2(\lfloor \log_q(3g+4) \rfloor + 2)$. The graph G contains at most $2m$ disjoint Π-homotopic cycles.

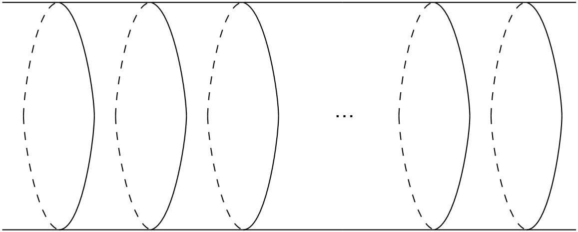

Figure: Disjoint homotopic cycles.

Homotopic: Let H be a Π_H-embedded graph and let C and C' be disjoint cycles of H, C and C' are Π_H-homotopic if they bound a cylinder in the embedding Π_H of H.

The disjoint Π-noncontractible cycles in (G, Π)

Proposition

Let H be a Π_H-embedded graph in a surface of genus g_H. If $C_1, \ldots, C_k$ are cycles of H that are disjoint, Π_H-noncontractible and pairwise Π_H-nonhomotopic, then

$$k \leq \begin{cases} g_H & \text{if } g_H \leq 1 \\ 3g_H - 3 & \text{if } g_H \geq 2 \end{cases}$$

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Corollary (H., Kawarabayashi)

Let $q = \frac{9073}{9072}$ and $m = 2(\lfloor \log_q(3g+4) \rfloor + 2)$. The graph G contains at most

$$2m \times (3g+3) = O(g \log g)$$

disjoint Π-noncontractible cycles.

A separator-based variant of disjoint nested cycles

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Proposition

Let $q = \frac{9073}{9072}$. Let C be a Π-contractible cycle of G and suppose that $A \subseteq V(C)$ separates the interior and the exterior of C with $|A| = k$. Then, $Int(C, \Pi)$ contains at most $2(\lfloor \log_q(60k + 180) \rfloor + 2)$ disjoint Π-nested cycles.

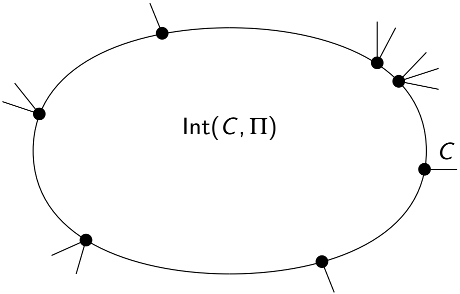

Figure: The cycle C and its separator.

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The bound for the disjoint nested cycles is:

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If $k << g$, then we get a significant improvement locally.

Proof strategy

Let G be an excluded minor for a surface S of genus g.

To bound the order of G, let's make use of the separator-based variant of disjoint nested cycles.

Goal: Find $G_1 \subseteq G$ such that:

- G_1 is a 2-connected Π-contractible subgraph of G,
- G_1 has a separator of size $O(\log^2 g)$,
- $|V(G)| \leq f(g) \times |V(G_1)|$ for some polynomial function f.

5. Bound on the order of G

Proof outline

1. Step 1: reduce from G to a subgraph $G_0 \subseteq G$ that has a separator A in G and so that the bridges on A in G_0 are Π-contractible
2. Step 2: reduce from G_0 to a 2-connected Π-contractible subgraph $G_1 \subseteq G_0$ with a $O(\log^2 g)$ separator
3. Step 3: show that the order of G_1 is sub-polynomial in g

Bridge: A bridge B on a vertex set A_H in a graph H is a connected component of $H - A_H$ together with its neighbors in A_H.

Step 1: reduce from G to a subgraph $G_0 \subseteq G$ that has a separator A in G and so that the bridges on A in G_0 are Π-contractible

Balanced tree decomposition of G

We use the balanced separator theorem from Robertson and Seymour (RS86)

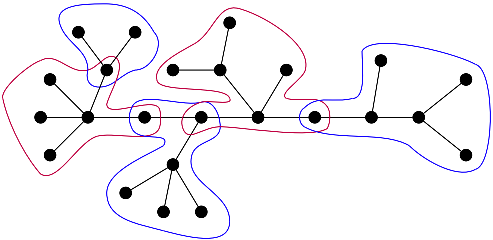
Balanced tree decomposition of G

We use the balanced separator theorem from Robertson and Seymour (RS86) to show the following:

Lemma

Let H be a graph and let $(T, (V_t)_{t \in T})$ be a tree decomposition of H. For any $k \geq 1$, there is a sequence of subtrees $(T_1, \ldots, T_k)$ of T such that

1. *for every $1 \leq i, j \leq k$, $|\bigcup_{t \in T_i} V_t| \leq 3|\bigcup_{t \in T_j} V_t|$,*
2. *for every $1 \leq i \leq k$, $V(T_i) \cap (\bigcup_{1 \leq j \neq i \leq k} V(T_j))$ has size at most $\lfloor \log_{\frac{4}{3}} 3k \rfloor$.*



Reducing from G to $G_0 \subseteq G$

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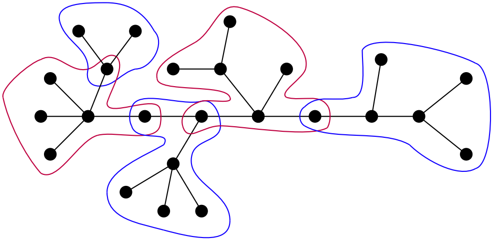

Figure: A tree decomposition with a balanced separation

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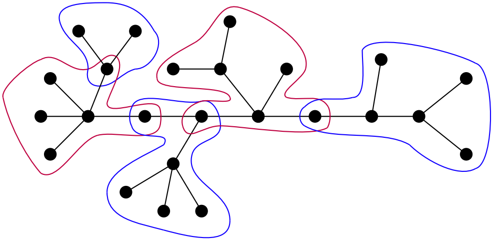

Figure: A tree decomposition with a balanced separation

Using the $2m \times (3g+3)$ bound on the number of disjoint Π-noncontractible cycles in (G, Π),

Reducing from G to $G_0 \subseteq G$

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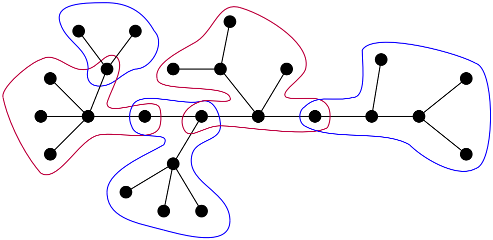

Figure: A tree decomposition with a balanced separation

Using the $2m \times (3g+3)$ bound on the number of disjoint Π-noncontractible cycles in (G, Π), we get that all the bridges in one of the parts are Π-contractible.

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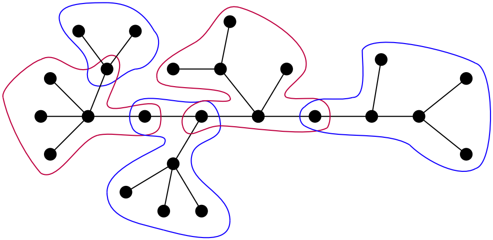

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Let G_0 be this part.

Reducing from G to $G_0 \subseteq G$

We found:

- G_0 has a separator A of size $O(g \log^2 g)$,
- The bridges on A in G_0 are Π-contractible,
- $|V(G)| \leq f(g) \times |V(G_0)|$ with $f(g) = \tilde{O}(g)$.

Step 2: reduce from G_0 to a Π-contractible subgraph $G_1 \subseteq G_0$ with a $O(\log^2 g)$ separator

The treewidth of the bridges on A in G_0

Robertson, Seymour, and Thomas (RST94) showed the following:

Theorem

Every planar graph with no $k \times k$ grid minor has treewidth at most $6k - 5$.

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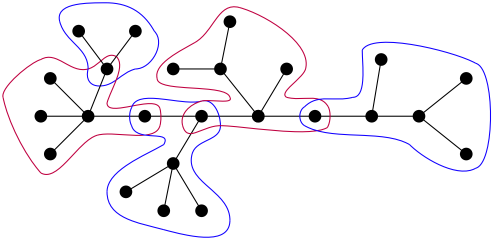
→ Each bridge has treewidth at most $12m + 7$ by the result stated above.

Reducing from G_0 to $G_1 \subseteq G_0$

Take a tree decomposition of one of the bridges of width at most $12m + 7$ and divide it into $|A| + 1 = O(g \log^2 g)$ balanced parts.



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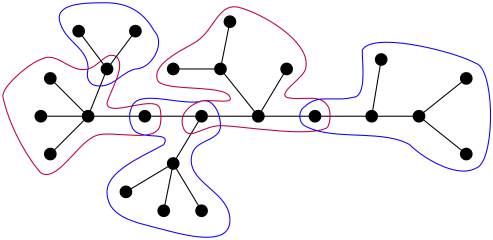

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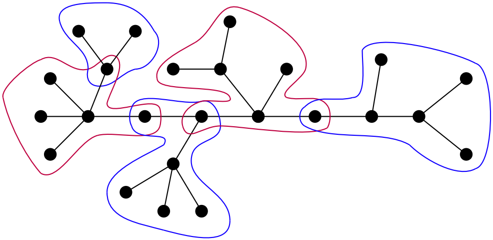

Figure: A tree decomposition with a balanced separation

By the pigeon hole principle, one of the parts (minus its intersection with the other parts) contain no vertex from A.

Let G_1 be the largest 2-connected component of this part.

Reducing from G_0 to $G_1 \subseteq G_0$ with a $O(\log^2 g)$ separator

We found:

- G_1 is Π-contractible and 2-connected,
- G_1 has a separator of size $O(\log^2 g)$,
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→ Remark that, because G_1 has a separator of size $O(\log^2 g)$, then G_1 contains at most $O(\log \log g)$ nested cycles (by the separator-based variant)

Step 3: show that the order of G_1 is sub-polynomial in g

Showing that the order of G_1 is sub-polynomial in g

- Step 3.1: bound the maximum degree in G_1 and the maximum size of faces in (G_1, Π)
- Step 3.2: bound the order of G_1

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If G_1 has a large degree vertex:

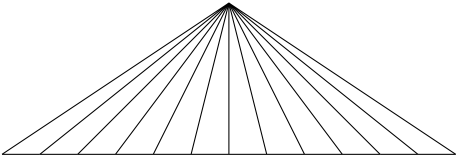

Figure: Faces on a large degree vertex.

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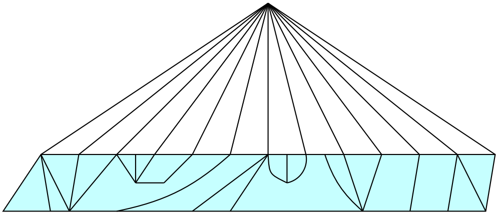

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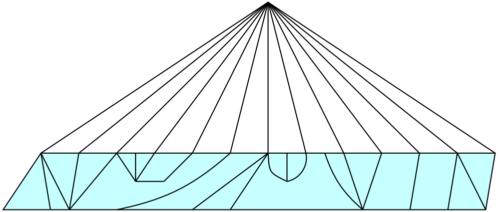

Figure: Faces on a large degree vertex.

Make use of the $O(\log \log g)$ bound on the number of Π-nested cycles.

Step 3.2: bound the order of G_1

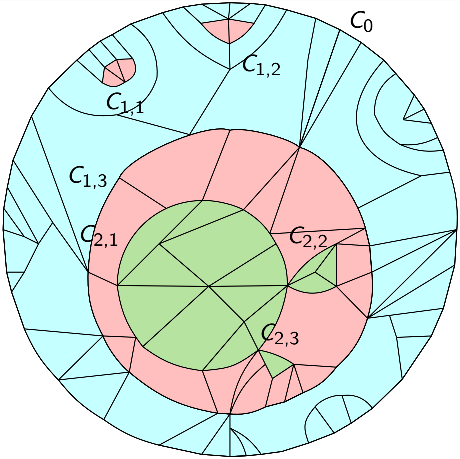

Figure: Partition into radius classes with respect to C_0.

Radius (of a face): Distance of this face to the outer face.

6. Conclusion

Conclusion: From a double-exponential to a polynomial bound

Theorem (Seymour 1993)

Let S be a given surface of genus g, every excluded minor for S has at most 2^{2^k} vertices where $k = (3g+9)^9$.

→ From a double-exponential bound...

Theorem (H., Kawarabayashi)

Let S be a given surface of Euler genus g. Every excluded minor for S has $O(g^{2+\varepsilon})$ vertices for every $\varepsilon > 0$.

→ ... to an almost quadratic bound

Future work

We conjecture that the optimal bound is almost linear.

Open problem

Is there an almost linear bound on the order of G?

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Another promising research direction is to make use of our new results together with computational power to determine the exact list of excluded minors for the torus and the Klein bottle.