A quasi-polynomial bound for the excluded minors for a surface

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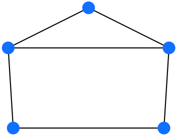
1. Introduction

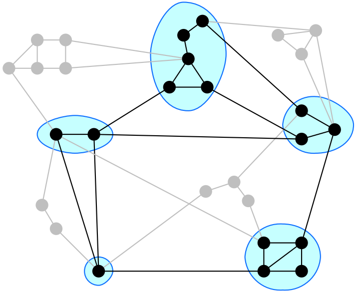
Definition of minor

Definition (Minor)

A minor H of a graph G can be obtained from G by a series of vertex deletions, edge deletions and edge contractions.



(a) The house graph H



(b) Model of H in a graph G

Figure: Minor

Definition of surface and embedding

Examples of surfaces: Sphere (g=0), torus (g=2), double-torus (g=4), projective plane (g=1), Klein bottle (g=2)...

Embedding (informal definition): An embedding Π of a graph G on a surface S is a drawing of G on S without crossings.

Genus: Measure of the complexity of a surface (Euler genus)

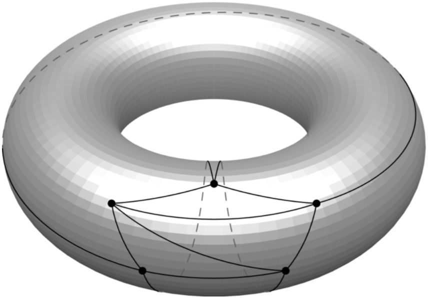


Figure: An embedding of K_5 on the torus

Family of graphs closed under minors

Definition (Closed under minors)

A family of graphs $\mathcal{C}$ is closed under minors if, for every $G \in \mathcal{C}$ and H minor of G, we have $H \in \mathcal{C}$.

Definition (Excluded minor)

Let $\mathcal{C}$ be a class of graphs closed under minors. An excluded minor for the class $\mathcal{C}$ is a graph $G \notin \mathcal{C}$ so that every proper minor of G is in $\mathcal{C}$.

Notice that: *G is minimal so that $G \notin \mathcal{C}$*

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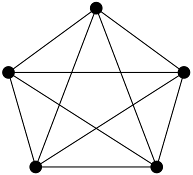
Theorem (Graph Minor Theorem, Robertson & Seymour [4])

Every family of graphs that is closed under minors can be defined by a finite set of excluded minors.

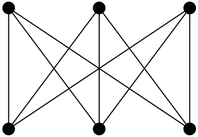
The Graph Minor Theorem applied to graphs on surfaces

Theorem (Wagner, also corollary of the GMT)

A graph is planar if and only if it does not contain K_5 or $K_{3,3}$ as its minor.



(a) K_5



(b) $K_{3,3}$

Figure: The excluded minors for the sphere: K_5 and $K_{3,3}$

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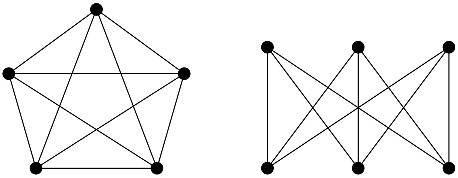

(a) K_5 (b) $K_{3,3}$

Figure: The excluded minors for the sphere: K_5 and $K_{3,3}$

Corollary (Robertson & Seymour [3])

Let S be a surface. Let $\mathcal{C}_S$ be the class of graphs that can be embedded on S without crossings. Then $\mathcal{C}_S$ can be defined by a finite set of excluded minors.

A bound on the size of these excluded minors

We know that there are a bounded number of excluded minors for a given surface, but we don't know how many or how big they are.

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Theorem (Seymour 1993 [5])

Let S be a given surface of genus g, every excluded minor for S has at most 2^{2^k} vertices where $k = (3g + 9)^9$.

Main result: a quasi-polynomial bound

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Let S be a given surface of Euler genus g. Every excluded minor for S has at most $U(g) = g^{O(\log^3 g)}$ vertices.

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Conjecture

Let S be a given surface of genus g, every excluded minor for S has a number of vertices polynomial in g.

2. Preliminary results

Genus of a graph

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The genus of a graph G is the genus of the smallest surface in which G can be embedded.

Why is it well defined?

- Let's show that there is a surface in which G can be embedded.
- Moreover, if G is embedded in a surface S of genus g, it can be embedded in any surface of genus $> g$.

Excluded minor for a surface

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- G can be embedded in a surface S' of genus $g+1$ or $g+2$, say with embedding Π.

  Let $e \in E(G)$, embed $G-e$ in the surface S with embedding Π_{G-e}. Then, adding the edge e to the embedding Π_{G-e} (in any way) create an embedding Π in a surface of genus $g+1$ or $g+2$.

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- G is 2-connected.

  Otherwise, decompose it into its 2-connected blocks.

Lemma

Let $G_1, \ldots, G_p$ ($p \geq 1$) be the 2-connected blocks of G. Then, for $1 \leq i \leq p$, G_i is an excluded minor for some surface S_i.

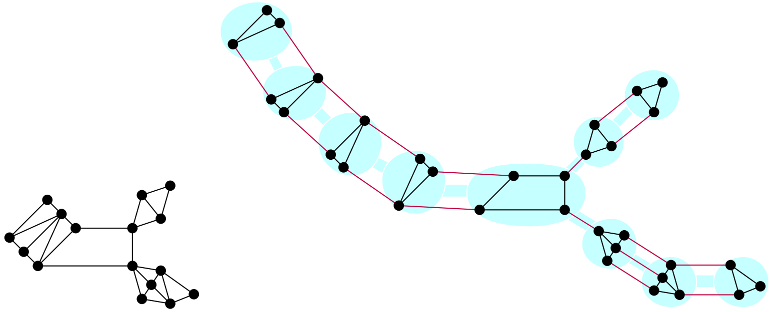
3. Main idea : using treewidth

Tree decomposition

Definition (Tree decomposition)

A tree decomposition of a graph G is a pair $(T, (V_t)_{t \in V(T)})$ with T a tree and, for every $t \in V(T)$, $V_t \subseteq V(G)$ with the following properties:

- $\forall v \in V(G), \{t \in V(T), v \in V_t\}$ *is a (non empty) tree,*
- $\forall e = uv \in E(G), \exists t \in V(T)$ *so that* $u, v \in V_t$.



(a) A graph G (b) A tree decomposition of G

Figure: Tree decomposition of a graph

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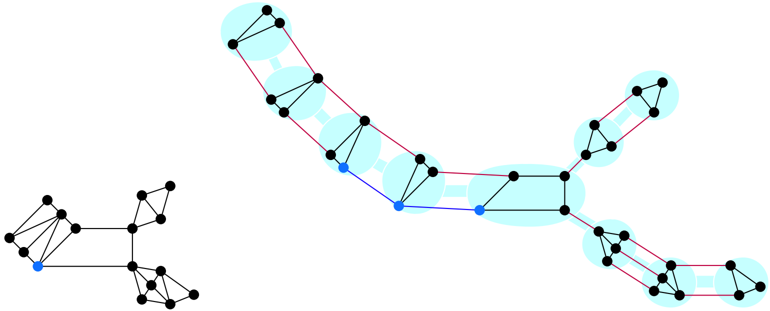

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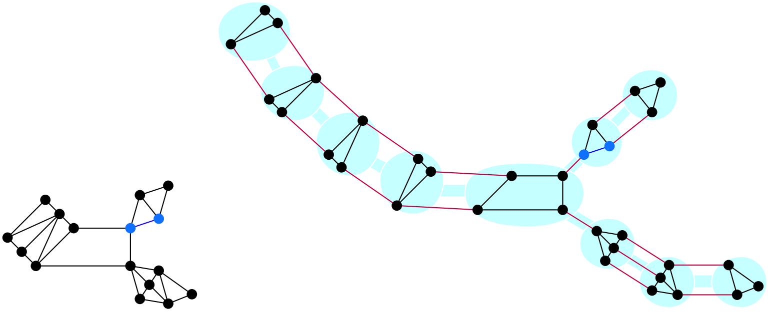

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Treewidth

The treewidth is a graph parameter that measures how close a graph is to a tree.

Definition (Width and treewidth)

The width of $(T, (V_t)_{t \in V(T)})$ *of* G *is* $\max_{t \in V(T)} |V_t| - 1$ *and the treewidth of* G *is the minimal width of its tree decompositions.*

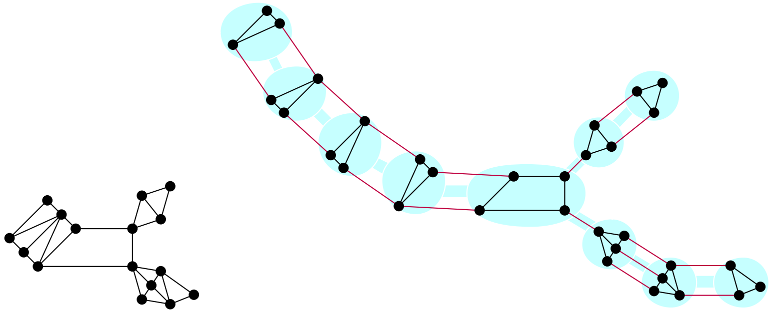

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Figure: Optimal tree decomposition of G: $tw(G) = 3$

Treewidth

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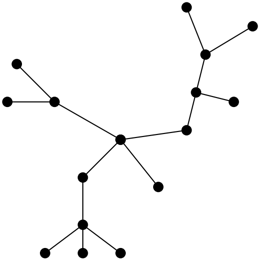
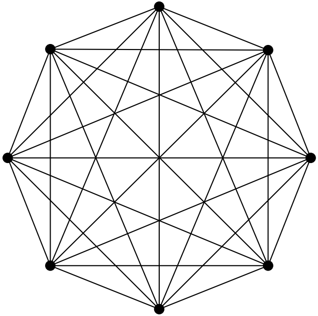


(a) A tree T: $tw(T) = 1$



(b) The clique K_8: $tw(K_8) = 7$

Figure: Examples for treewidth

Treewidth and graphs on surfaces

Planar graphs have unbounded treewidth.

Lemma (Treewidth of a grid)

For $k \geq 1$, the $k \times k$ grid has treewidth k.

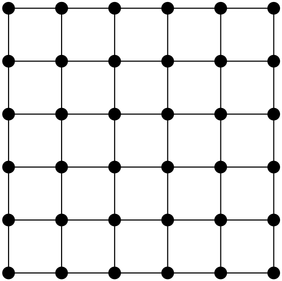


Figure: The 6×6 grid has treewidth 6.

Therefore, for a surface S, graphs embeddable on S have unbounded treewidth.

Known results on tree decompositions of G

Theorem (Seymour [5, (3.3)])

The treewidth of G is bounded by a polynomial in g:

$$tw(G) \leq T(g)$$

with $T(g) = 3(g+3)^2(3g+16) - 3 = O(g^3)$

Theorem (Seymour [5, claim (5) in (4.1)])

Let $(T, (V_t)_{t \in T})$ be a (special) tree decomposition of G of width $< w$. Then, the maximum degree of T is bounded by a polynomial in g and w:

$$\Delta(T) \leq \Delta_T(g, w)$$

with $\Delta_T(g, w) = 2g + 2w$

Improvement on the bounds for tree decomposition

Proposition (H., Kawarabayashi 2025+)

The treewidth of G is bounded by the following function of g:

$$tw(G) \leq T(g)$$

with $T(g) = 264(g+2)(m+1) - 1 = O(g \log g)$*, where* $m = 2(\lfloor \log_q(3g+4) \rfloor + 2)$ *and* $q = \frac{1153}{1152}$.

Corollary (H., Kawarabayashi 2025+)

Let $(T, (V_t)_{t \in T})$ *be a (special) tree decomposition of G of width* $tw(G)$*. Then, the degree of T is bounded by a polynomial in g:*

$$\Delta(T) \leq \Delta_T(g)$$

with $\Delta_T(g) = \Delta_T(g, T(g)+1) = 2g + 2(T(g)+1) = O(g \log g)$.

Proof strategy

Let $(T, (V_t)_{t\in T})$ be a (special) tree decomposition of G.
To bound the order of G, it suffices to find a bound on the height of T.

- Treewidth of G: $O(g \log g)$
- Maximum degree of T: $O(g \log g)$
- Height of T: ??

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Goal: To obtain $|V(G)| = g^{O(\log^3 g)}$, bound the height of T by $O(\log^3 g)$.

4. Height of a tree decomposition of G

Proof outline

1. Step 1: reduce to planar graphs
2. Step 2: prove the bound
3. Trick: use pathwidth

Step 1: reduce to planar graphs

Contractible cycles

Let H be a Π_H-embedded graph in a surface S_H.

Definition (Contractible cycle)

Let C be a cycle of H, C is Π_H-contractible if C bounds a disk in the embedding Π_H of H.

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Corollary (H., Kawarabayashi 2025+)

Let $q = \frac{1153}{1152}$ and $m = 2(\lfloor \log_q(3g+4) \rfloor + 2)$. G contains at most $2m \times (3g+3) = O(g \log g)$ disjoint Π-noncontractible cycles.

Bounding the height of a tree decomposition of G

Suppose that we have the following result:

Proposition (H., Kawarabayashi 2025+)

Let $(T, (V_t)_{t\in T})$ be a (special) tree decomposition of G of width w. Let P be a path from t_1 to t_2 of length $P(g, w)$ in T. Let $G_0 = \bigcup_{t\in \overline{P}} V_t - (V_{t_1} \cup V_{t_2})$. Then $\Pi(G_0)$ is not an embedding in a disk on S.

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Then, we can show:

Theorem (H., Kawarabayashi 2025+)

Let $(T, (V_t)_{t\in T})$ be a (special) tree decomposition of G of width w. Then, T contains no path of length more than $P'(g, w) = (2m(3g + 3) + 1) \times P(g, w) - 1$.

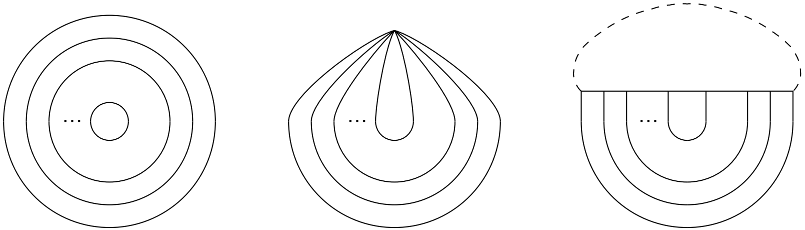
Proof outline: Proceed by contradiction: there is a path of length $> P'(g, w)$. Cut this path into paths of length $\geq P(g, w)$, there are at least $2m(3g + 3) + 1$ of them. Contradiction.

Step 2: prove the bound

Main structural result: Well-nested cycles

Proposition (H., Kawarabayashi 2025+)

Let $q = \frac{1153}{1152}$ *and* $m = 2(\lfloor \log_q(3g+4) \rfloor + 2)$*. The graph* G *contains at most* m *cycles that are* Π*-well-nested.*



(a) Fully well-nested cycles

(b) Well-nested cycles pinched on a vertex

(c) Well-nested cycles pinched on a face

Figure: Well-nested cycles. The solid lines indicate paths, whereas the dotted lines show the boundaries of the faces which the isolated paths use.

Definition of a face

Definition (Face (informal))

The faces of an embedding Π of G are found by following a walk from a side of an edge, until it comes back to this same side of the edge.
The size of a face is the length of the corresponding walk.

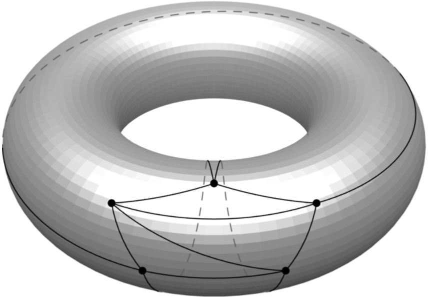


Figure: An embedding of K_5 on the torus

Bounding the degree of G and the maximum size of a face of (G, Π)

Theorem (H., Kawarabayashi 2025+)

Let $\tilde{g} = 4(6g+7)$, $q = \frac{1153}{1152}$ *and* $m = 2(\lfloor \log_q(3g+4) \rfloor + 2)$.

$$\Delta(G) \leq \Delta(g) \quad \textit{and} \quad \Delta_F(G, \Pi) \leq \Delta(g)$$

with $\Delta(g) = 2m(\tilde{g}+1)^4 \left(4m(\tilde{g}+1)^2\right)^{m^2} = g^{O(\log^2 g)}$

Bounding the degree of G and the maximum size of a face of (G, Π)

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with $\Delta(g) = 2m(\tilde{g}+1)^4 \left(4m(\tilde{g}+1)^2\right)^{m^2} = g^{O(\log^2 g)}$

Proof outline: Prove by induction that G contains $m+1$ Π-well-nested cycles. Contradiction.

Bounding the height of a tree decomposition of G

Proposition (H., Kawarabayashi 2025+)

Let $(T, (V_t)_{t \in T})$ be a (special) tree decomposition of G of width w. Let P be a path from t_1 to t_2 of length $P(g, w)$ in T with

$$P(g, w) = \frac{\Delta(g)(\Delta(g)^{2m} - 1)}{\Delta(g) - 1} \times 2w + w + 2 = g^{O(\log^3 g)} \times O(w)$$

Let $G_0 = \bigcup_{t \in \overline{P}} V_t - (V_{t_1} \cup V_{t_2})$. Then $\Pi(G_0)$ is not an embedding in a disk on S.

Proof outline: Proceed by contradiction: G_0 is in a disk on S. Use the bound on the number of nested cycles and the separators given by the tree decomposition to prove a bound on the number of vertices of G_0.

Proof

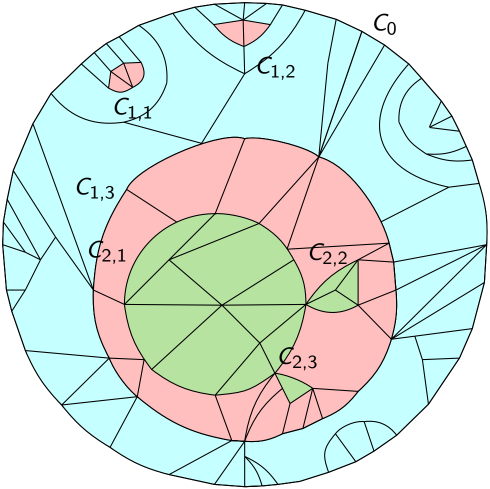

Figure: Partition into radius classes with respect to C_0.

A quasi single-exponential bound for G

Recap:

- Treewidth of G: $T(g) = O(g \log g)$
- Maximum degree of T: $\Delta(g) = O(g \log g)$
- Height of T: $P'(g, T(g)) = g^{O(\log^3 g)}$

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Corollary (H., Kawarabayashi 2025+)

Let G be an excluded minor for a surface S' of genus g.

$$|V(G)| \leq 2^{Q(g)}$$

with $Q(g)$ a quasi-polynomial in g so that

$$Q(g) = g^{O(\log^3 g)}$$

Trick: use pathwidth

From a quasi single-exponential to a quasi polynomial bound: pathwidth

Proposition (Bodlaender [1])

Let G be a graph, then

$$pw(G) = O(tw(G)\log(|V(G)|))$$

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Corollary (H., Kawarabayashi 2025+)

Let G be an excluded minor for a surface S of genus g. There exists a constant A so that

$$pw(G) \leq R(g) = A \times T(g) \times Q(g)$$

with $T(g) = O(g \log g)$ *and* $Q(g) = g^{O(\log^3 g)}$.

A quasi-polynomial bound

Corollary (H., Kawarabayashi 2025+)

Let G be an excluded minor for a surface S of genus g. There exists a constant A so that

$$|V(G)| \leq A \times S(g)$$

with $S(g) = P'(g, R(g)) \times T(g) \times Q(g) = g^{O(\log^3 g)}$.

Proof outline: Use the bound on the pathwidth and use again the bound on the height of the tree in the tree decomposition (= size of the path).

5. Conclusion

Conclusion: From a double-exponential to a polynomial bound

Theorem (Seymour 1993 [5])

Let S be a given surface of Euler genus g. Every excluded minor for S has at most 2^{2^k} vertices where $k = (3g+9)^9$.

Theorem (H., Kawarabayashi 2025+)

Let S be a given surface of Euler genus g. Every excluded minor for S has at most $U(g) = g^{O(\log^3 g)}$ vertices.

Conclusion: Subsidiary results

- Forbidden structures: nested cycles (but also isolated paths and homotopic cycles)
- Treewidth:

$$O(g^3) \rightarrow O(g \log g)$$

- Maximum degree of the tree of an optimal tree decomposition of G:

$$O(g^3) \rightarrow O(g \log g)$$

- Maximum size of a subdivision of a grid in G:

$$O(g^{3/2}) \rightarrow O(\sqrt{g} \log g)$$

Future work

We are currently pursuing research in order to show a polynomial bound on the order of G.

Conjecture

Let S be a given surface of genus g, every excluded minor for S has a number of vertices polynomial in g.

Thank you for your attention

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