

# A quasi-polynomial bound for the excluded minors for a surface

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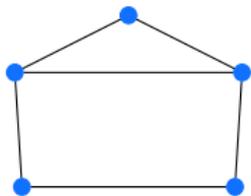
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# 1. Introduction

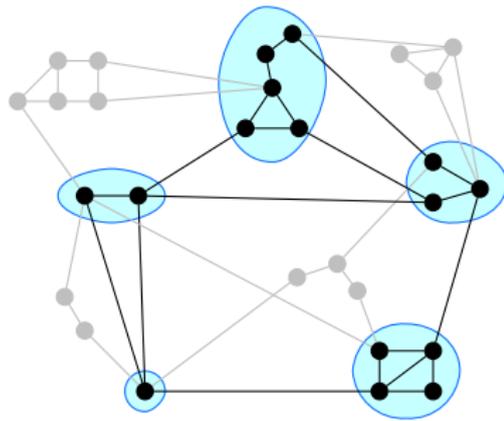
## Definition of minor

### Definition (Minor)

*A minor  $H$  of a graph  $G$  can be obtained from  $G$  by a series of vertex deletions, edge deletions and edge contractions.*



(a) The house graph  $H$



(b) Model of  $H$  in a graph  $G$

Figure: Minor

## Definition of surface and embedding

**Examples of surfaces:** Sphere ( $g=0$ ), torus ( $g=2$ ), double-torus ( $g=4$ ), projective plane ( $g=1$ ), Klein bottle ( $g=2$ )...

**Embedding (informal definition):** An embedding  $\Pi$  of a graph  $G$  on a surface  $S$  is a drawing of  $G$  on  $S$  without crossings.

**Genus:** Measure of the complexity of a surface (Euler genus)

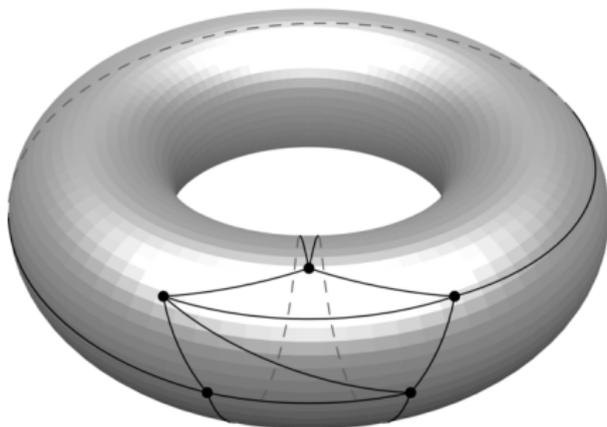


Figure: An embedding of  $K_5$  on the torus

## Family of graphs closed under minors

### Definition (Closed under minors)

*A family of graphs  $\mathcal{C}$  is closed under minors if, for every  $G \in \mathcal{C}$  and  $H$  minor of  $G$ , we have  $H \in \mathcal{C}$ .*

### Definition (Excluded minor)

*Let  $\mathcal{C}$  be a class of graphs closed under minors. An excluded minor for the class  $\mathcal{C}$  is a graph  $G \notin \mathcal{C}$  so that every proper minor of  $G$  is in  $\mathcal{C}$ .*

*Notice that:  $G$  is minimal so that  $G \notin \mathcal{C}$*

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### Theorem (Graph Minor Theorem, Robertson & Seymour 2004)

*Every family of graphs that is closed under minors can be defined by a finite set of excluded minors.*

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Height of a tree decomposition of  $G$

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# The Graph Minor Theorem for graphs on surfaces

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### Lemma

*Let  $S$  be a surface. Let  $\mathcal{C}_S$  be the class of graphs that can be embedded on  $S$  without crossings. Then  $\mathcal{C}_S$  is closed under minors.*

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### Theorem (Wagner, also corollary of the GMT)

A graph is planar if and only if it does not contain  $K_5$  or  $K_{3,3}$  as its minor.

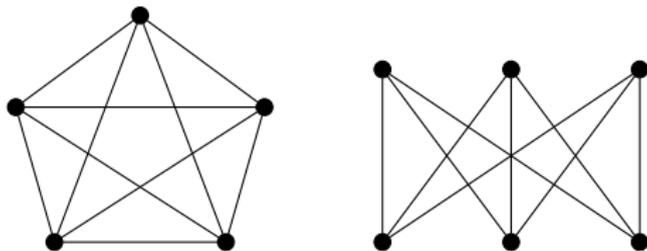


Figure: The excluded minors for the sphere:  $K_5$  and  $K_{3,3}$

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### Theorem (Seymour 1993)

*Let  $S$  be a given surface of genus  $g$ , every excluded minor for  $S$  has at most  $2^{2^k}$  vertices where  $k = (3g + 9)^9$ .*

## A bound on the excluded minors for a surface

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### Conjecture

*Let  $S$  be a given surface of genus  $g$ , every excluded minor for  $S$  has a number of vertices polynomial in  $g$ .*

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- Treewidth:

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- Finer structural results (forbidden structures): nested cycles and homotopic cycles

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## Algorithmic implications

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There are several algorithms that relies on bounds for the excluded minors for surfaces:

- Membership test for graphs on surfaces (FL89, AGK08)  
*order* or *treewidth*
- Embedding graphs in a given surface (KMR08) *order*
- Computing a graph minor decomposition (GKR13) *order*

## 2. Preliminary results

## Genus of a graph

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*The genus of a graph  $G$  is the genus of the smallest surface in which  $G$  can be embedded.*

- It is well defined: There is always a surface in which  $G$  can be embedded.
- It is minor-monotone: if  $H$  is a minor of  $G$ , then  $g(H) \leq g(G)$ .

## Excluded minor for a surface

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- $G$  can be embedded in a surface  $S'$  of genus  $g + 1$  or  $g + 2$ , say with embedding  $\Pi$ .

Let  $e \in E(G)$ , embed  $G - e$  in the surface  $S$  with embedding  $\Pi_{G-e}$ . Then, adding the edge  $e$  to the embedding  $\Pi_{G-e}$  (in any way) create an embedding  $\Pi$  in a surface of genus  $g + 1$  or  $g + 2$ .

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- $G$  is 2-connected.

Otherwise, decompose it into its 2-connected blocks.

### Lemma

Let  $G_1, \dots, G_p$  ( $p \geq 1$ ) be the 2-connected blocks of  $G$ . Then, for  $1 \leq i \leq p$ ,  $G_i$  is an excluded minor for some surface  $S_i$ .

### Lemma

Let  $G_1, \dots, G_p$  ( $p \geq 1$ ) be the 2-connected blocks of  $G$ . Then,  
 $g(G) = g(G_1) + \dots + g(G_p)$ .

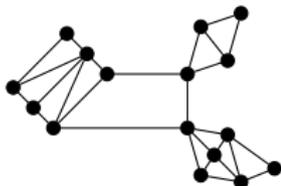
### 3. Main idea : using treewidth

# Tree decomposition

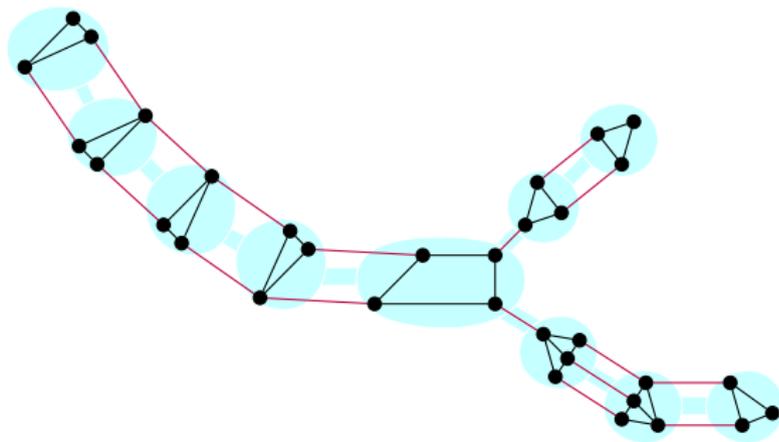
## Definition (Tree decomposition)

A tree decomposition of a graph  $G$  is a pair  $(T, (V_t)_{t \in V(T)})$  with  $T$  a tree and, for every  $t \in V(T)$ ,  $V_t \subseteq V(G)$  with the following properties:

- $\forall v \in V(G), \{t \in V(T), v \in V_t\}$  is a (non empty) tree,
- $\forall e = uv \in E(G), \exists t \in V(T)$  so that  $u, v \in V_t$ .



(a) A graph  $G$



(b) A tree decomposition of  $G$

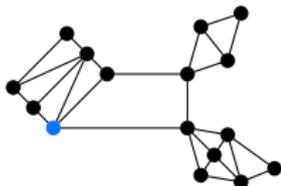
Figure: Tree decomposition of a graph

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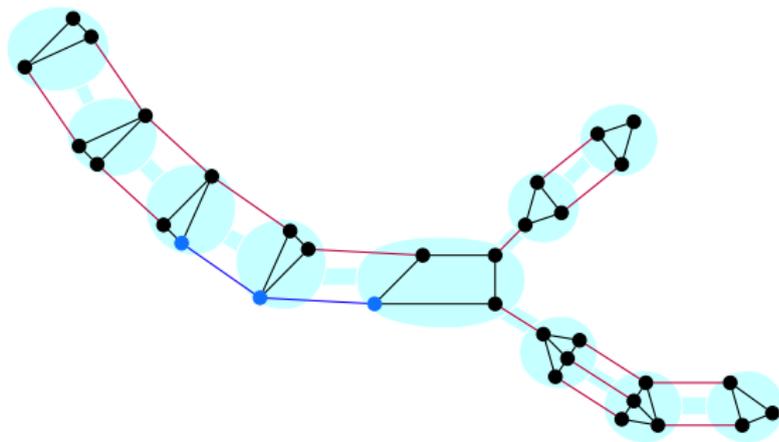
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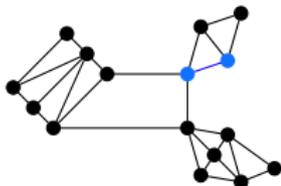
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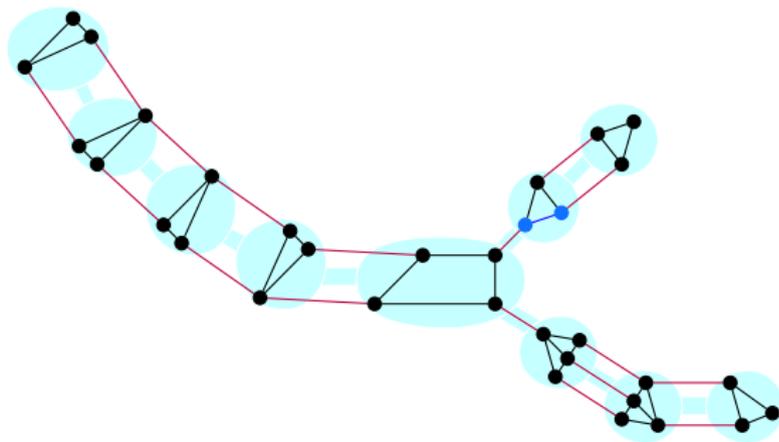
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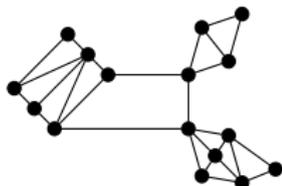
Figure: Tree decomposition of a graph

# Treewidth

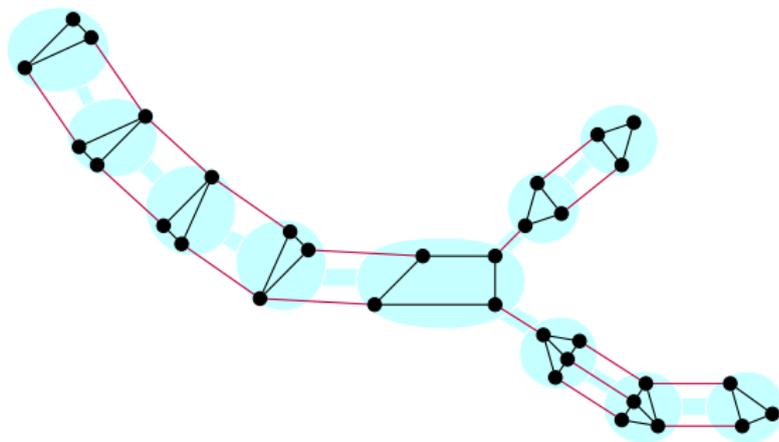
The treewidth is a graph parameter that measures how close a graph is to a tree.

## Definition (Width and treewidth)

The width of  $(T, (V_t)_{t \in V(T)})$  of  $G$  is  $\max_{t \in V(T)} |V_t| - 1$  and the treewidth of  $G$  is the minimal width of its tree decompositions.



(a) A graph  $G$



(b) A tree decomposition of  $G$

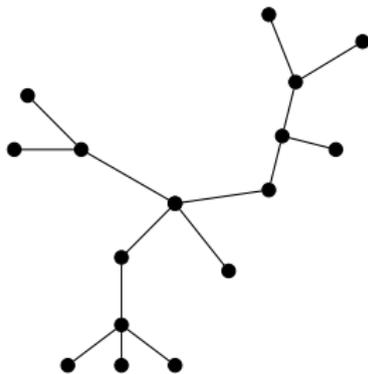
Figure: Optimal tree decomposition of  $G$ :  $tw(G) = 3$

# Treewidth

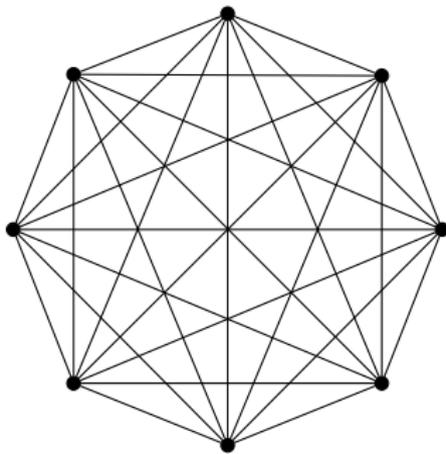
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(a) A tree  $T$ :  $tw(T) = 1$



(b) The clique  $K_8$ :  $tw(K_8) = 7$

Figure: Examples for treewidth

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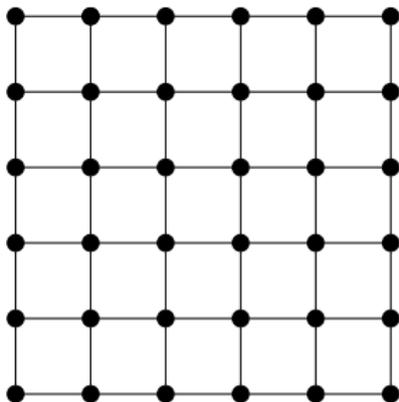
# Treewidth and graphs on surfaces

## Treewidth and graphs on surfaces

Planar graphs have unbounded treewidth.

### Lemma (Treewidth of a grid)

*For  $k \geq 1$ , the  $k \times k$  grid has treewidth  $k$ .*



**Figure:** The  $6 \times 6$  grid has treewidth 6.

Therefore, for a surface  $S$ , graphs embeddable on  $S$  have unbounded treewidth.

## Tree decompositions of excluded minors for surfaces

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Treewidth of  $G$ :

$$\begin{array}{cc} O(g^3) & \text{Seymour 1993} \\ \downarrow & \\ O(g \log g) & \text{H., Kawarabayashi} \end{array}$$

Max degree of the tree of a linked tree decomposition of width  $tw(G)$  of  $G$ :

$$\begin{array}{cc} O(g^3) & \text{Seymour 1993} \\ \downarrow & \\ O(g \log g) & \text{H., Kawarabayashi} \end{array}$$

## Proof strategy

Let  $G$  be an excluded minor for a surface  $S$  of genus  $g$ .

Let  $(T, (V_t)_{t \in T})$  be a linked tree decomposition of  $G$  of width  $tw(G)$ .

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To bound the order of  $G$ , we will use the tree decomposition.

- Treewidth of  $G$ :  $O(g \log g)$
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- Maximum degree of  $T$ :  $O(g \log g)$
- Height of  $T$ : ??

Goal: Bound the height of  $T$ .

## 4. Height of a tree decomposition of $G$

## Proof outline

- 1 Step 1: reduce to planar graphs
- 2 Step 2: prove the bound
- 3 Trick: use pathwidth

## Step 1: reduce to planar graphs

## Main structural result: Disjoint nested cycles

$G$  has an embedding  $\Pi$  in a surface  $S'$  of genus  $g + 1$  or  $g + 2$ .

Proposition (H., Kawarabayashi)

Let  $q = \frac{1153}{1152}$  and  $m = 2(\lfloor \log_q(3g + 4) \rfloor + 2)$ . The graph  $G$  contains at most  $m$  disjoint  $\Pi$ -nested cycles.

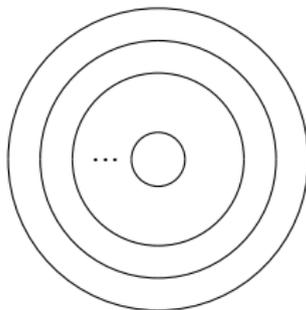


Figure: Disjoint nested cycles.

**Contractible:** Let  $H$  be a  $\Pi_H$ -embedded graph and  $C$  be a cycle of  $H$ ,  $C$  is  $\Pi_H$ -contractible if  $C$  bounds a disk in the embedding  $\Pi_H$  of  $H$ .

## Main structural result: Disjoint homotopic cycles

### Proposition (H., Kawarabayashi)

Let  $q = \frac{1153}{1152}$  and  $m = 2(\lfloor \log_q(3g + 4) \rfloor + 2)$ . The graph  $G$  contains at most  $2m$  disjoint  $\Pi$ -homotopic cycles.

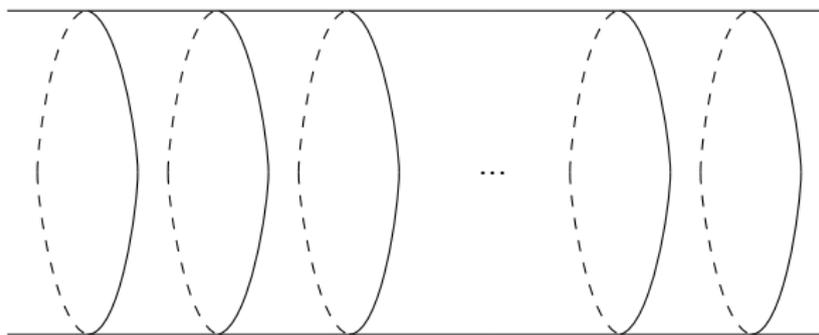


Figure: Disjoint homotopic cycles.

**Homotopic:** Let  $H$  be a  $\Pi_H$ -embedded graph and let  $C$  and  $C'$  be disjoint cycles of  $H$ .  $C$  and  $C'$  are  $\Pi_H$ -homotopic if they bound a cylinder in the embedding  $\Pi_H$  of  $H$ .

## The disjoint $\Pi$ -noncontractible cycles in $(G, \Pi)$

### Proposition

Let  $H$  be a  $\Pi_H$ -embedded graph in a surface of genus  $g_H$ . If  $C_1, \dots, C_k$  are cycles of  $H$  that are disjoint,  $\Pi_H$ -noncontractible and pairwise  $\Pi_H$ -nonhomotopic, then

$$k \leq \begin{cases} g_H & \text{if } g_H \leq 1 \\ 3g_H - 3 & \text{if } g_H \geq 2 \end{cases}$$

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### Corollary (H., Kawarabayashi)

Let  $q = \frac{1153}{1152}$  and  $m = 2(\lfloor \log_q(3g + 4) \rfloor + 2)$ . The graph  $G$  contains at most

$$2m \times (3g + 3) = O(g \log g)$$

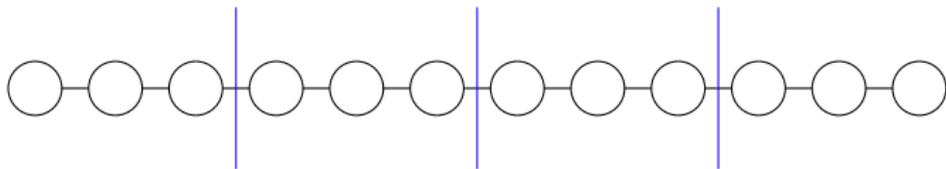
disjoint  $\Pi$ -noncontractible cycles.

## Reducing to a planar graph

Take the longest path  $P$  in the tree decomposition of  $G$  and divide it into  $2m \times (3g + 3) + 1$  equal-size subpaths.

## Reducing to a planar graph

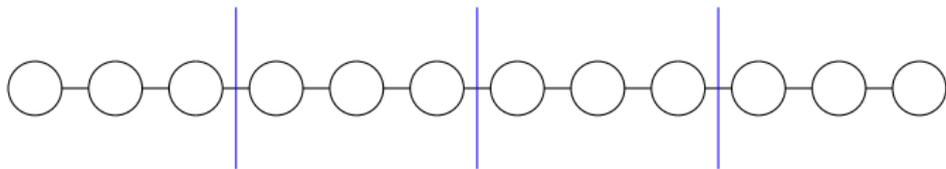
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**Figure:** The longest path  $P$  in the tree decomposition of  $G$ , divided into subpaths.

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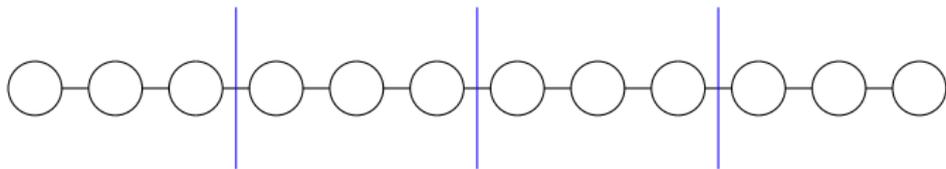


**Figure:** The longest path  $P$  in the tree decomposition of  $G$ , divided into subpaths.

Using the  $2m \times (3g + 3)$  bound on the number of disjoint  $\Pi$ -noncontractible cycles in  $(G, \Pi)$ ,

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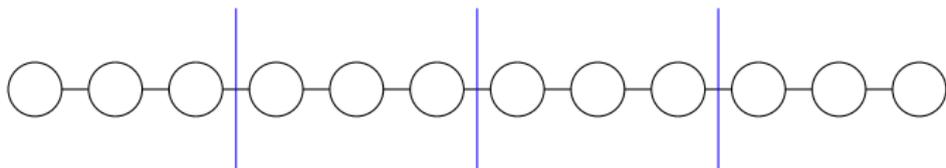


**Figure:** The longest path  $P$  in the tree decomposition of  $G$ , divided into subpaths.

Using the  $2m \times (3g + 3)$  bound on the number of disjoint  $\Pi$ -noncontractible cycles in  $(G, \Pi)$ , we get that the subgraph  $G_0$  induced by the interior of one of the subpaths is  $\Pi$ -contractible.

## Reducing to a planar graph

Take the longest path  $P$  in the tree decomposition of  $G$  and divide it into  $2m \times (3g + 3) + 1$  equal-size subpaths.



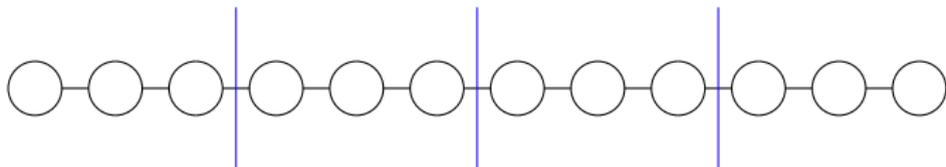
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Remark that it will automatically gives a bound on the size of the corresponding subpath and therefore on the size of  $P$ .

## Step 2: prove the bound

## Bounding the height of a tree decomposition of $G$

### Proposition (H., Kawarabayashi)

Let  $(T, (V_t)_{t \in T})$  be a linked tree decomposition of  $G$  of width  $w$ . Let  $P$  be a path from  $t_1$  to  $t_2$  of length  $P(g, w)$  in  $T$  and let  $G_0 = \bigcup_{t \in \bar{P}} V_t - (V_{t_1} \cup V_{t_2})$ . If  $G_0$  is an embedding in a disk of  $S$ , then

$$P(g, w) = g^{O(\log^3 g)} \times O(w)$$

**Proof outline:** Use the bound on the number of nested cycles and the separators given by the tree decomposition to prove a bound on the number of vertices of  $G_0$ .

## Proof

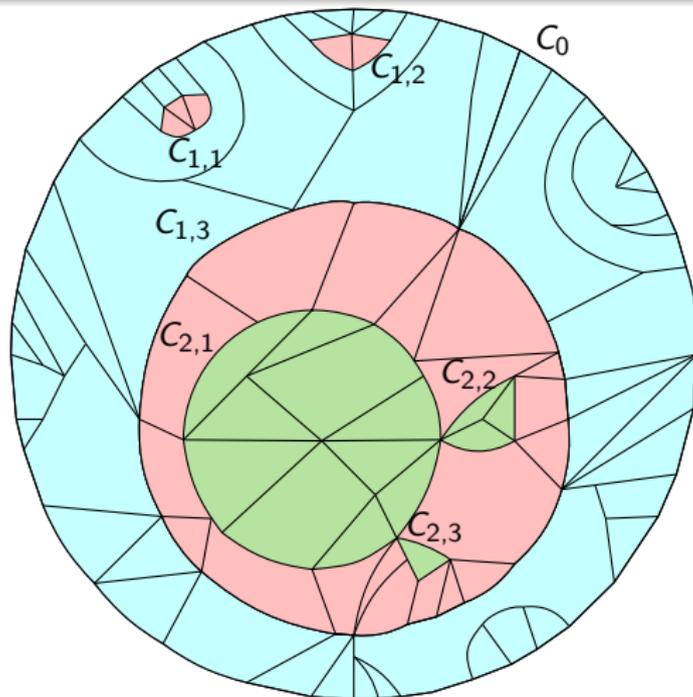


Figure: Partition into radius classes with respect to  $C_0$ .

**Radius (of a face):** Distance of this face to the outer face.

# A quasi single-exponential bound for $G$

## Recap:

- Treewidth of  $G$ :  $O(g \log g)$
- Maximum degree of  $T$ :  $O(g \log g)$
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## Corollary (H., Kawarabayashi)

Let  $G$  be an excluded minor for a surface  $S'$  of genus  $g$ .

$$|V(G)| \leq 2^{Q(g)}$$

with  $Q(g)$  a quasi-polynomial in  $g$  so that

$$Q(g) = g^{O(\log^3 g)}$$

Trick: use pathwidth

# From a quasi single-exponential to a quasi polynomial bound: pathwidth

## Proposition (Bodlaender 1998)

*Let  $H$  be a graph, then*

$$pw(H) = O(tw(H) \log(|V(H)|))$$

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Let  $G$  be an excluded minor for a surface  $S$  of genus  $g$ . There exists a constant  $A$  so that

$$pw(G) \leq R(g) = A \times T(g) \times Q(g)$$

with  $T(g) = O(g \log g)$  and  $Q(g) = g^{O(\log^3 g)}$ .

## A quasi-polynomial bound

### Corollary (H., Kawarabayashi)

Let  $G$  be an excluded minor for a surface  $S$  of genus  $g$ . There exists a constant  $A$  so that

$$|V(G)| \leq A \times S(g)$$

with  $S(g) = P'(g, R(g)) \times T(g) \times Q(g) = g^{O(\log^3 g)}$ .

**Proof outline:** Use the bound on the pathwidth and use again the bound on the height of the tree in the tree decomposition (= size of the path).

## 5. Conclusion

## Conclusion: From a double-exponential to a polynomial bound

### Theorem (Seymour 1993)

Let  $S$  be a given surface of genus  $g$ , every excluded minor for  $S$  has at most  $2^{2^k}$  vertices where  $k = (3g + 9)^9$ .

→ From a double-exponential bound...

### Theorem (H., Kawarabayashi)

Let  $S$  be a given surface of genus  $g$ . Every excluded minor for  $S$  has at most  $g^{O(\log^3 g)}$  vertices.

→ ... to a quasi-polynomial bound

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- Treewidth:

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- Finer structural results (forbidden structures): nested cycles and homotopic cycles

## Future work

We are currently pursuing research in order to show a polynomial bound on the order of  $G$ .

### Conjecture

*Let  $S$  be a given surface of genus  $g$ , every excluded minor for  $S$  has a number of vertices polynomial in  $g$ .*